Dynamics of BPS monopoles and dyons [1]

Dongsu Bak
University of Seoul
Seoul 130-742 Korea

Abstract

By an analysis of field equations of the SU(2) Yang-Mills-Higgs system, we obtain the effective field theory describing low energy interaction of BPS dyons and massless fields. This effective theory manifests electromagnetic duality and broken scale symmetry, and reproduces the multimonopole moduli space dynamics.

It is a fascination that non-singular magnetic monopoles arise as classical soliton solutions in certain spontaneously broken Yang-Mills gauge theories[1]. These monopoles are extended objects with definite mass and couple effectively in low energies to the electromagnetic fields. Recently, a number of exact results have been obtained in a class of supersymmetric gauge theories by exploiting the electromagnetic duality symmetry[2]. Magnetic monopoles relevant in this supersymmetric gauge theories are so called BPS monopoles[3]. In the BPS limit, there is a Bogomol'nyi bound on the static energy functional and we have degenerate static multimonopole solutions that saturate the bound. Originally this was a semiclassical result at most; but, in the supersymmetric gauge theories, Witten and Olive[4] showed that this result may continue to be valid even after quantum corrections are included.

To study the duality and other issues, various authors discussed the interaction of slowly moving BPS monopoles, mainly following the work of Manton[5]. The central point is that the moduli space of static N-monopole solutions is finite dimensional and possesses a metric coming from the kinetic energy terms of the Yang-Mill-Higgs Lagrangian. He suggested that low energy dynamics of a given set of monopoles and dyons may be approximated by geodesic motions on the moduli space. The metric for the two monopole moduli space was determined by Atiyah and Hitchin[6]. More recently[7], the knowledge on the metric has been used in theories with extended supersymmetry to show the existence of some of the dyonic states required by the electromagnetic duality conjecture of Montonen and Olive[8].

While Manton's approach is believed to give a valid approximate description, it deviates from the viewpoint of modern effective field theory; it is not based on all relevant degrees of freedom at low energy. Dynamical freedoms in Manton's approach are restricted to collective coordinates of monopoles, but the freedoms associated with photons and massless Higgs particles are also relevant at low energy. Instead of looking into the dynamics of collective coordinates of all monopoles, we will here obtain our effective field theory by studying how the collective coordinates of a single monopole get involved dynamically with soft electromagnetic and Higgs field excitations in the vicinity of the monopole. This effective theory can describe the low energy interaction of monopoles with on shell photons and Higgs particles, and in the appropriate limit produces the result of Manton as well. Moreover, it has distinctive advantage that underlying symmetries of the theory, the electromagnetic duality and spontaneously broken scale invariance, are clearly borne out.

First we shall recall the basic construct of the BPS dyon solution in an SU(2) gauge theory spontaneously broken to U(1). The Lagrangian density is ($a = 1, 2, 3$)

$$\mathcal{L} = -\frac{1}{4}G_a^{\mu\nu}G^a_{\mu\nu} - \frac{1}{2}(D_\mu\phi)_a(D^\mu\phi)_a \tag{1}$$

where

$$G_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + e\epsilon_{abc}A_b^\mu A_c^\nu, \tag{2}$$
$$(D_\mu\phi)_a = \partial_\mu\phi_a + e\epsilon_{abc}A_\mu^b\phi^c. \tag{3}$$

The field equations read

$$(D_\mu G^{\mu\nu})_a = -e\epsilon_{abc}(D^\nu\phi)^b\phi^c, \tag{4}$$
$$(D_\mu D^\mu\phi)_a = 0. \tag{5}$$

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Without any nontrivial Higgs potential in the Lagrangian density, this is a classically scale-invariant system. For this system, spontaneous symmetry breaking is achieved by demanding the asymptotic boundary condition

$$|\phi| = \sqrt{\phi_a \phi_a} \to f > 0, \quad \text{as } r \to \infty. \tag{6}$$

The unbroken U(1) will be identified with the electromagnetic gauge group below.

The above system admits static soliton solutions in the form of magnetic monopoles (or, more generally, dyons), the stability of which is derived from the topological argument. They will carry some nonzero charges with respect to long-ranged fields.

To obtain static solutions to field equations (4)and (5) with the lowest possible energy $M = f\sqrt{g^2+q^2}$ for given $g = \mp 4\pi n/e$ (n: positive integer) and $q = g\tan\beta$, it suffices to consider solutions to the first-order Bogomol'nyi equations[11]

$$B_i^a = \mp\cos\beta(D_i\phi)^a, \quad E_i^a = \mp\sin\beta(D_i\phi)^a, \ (D_0\phi)^a = 0. \tag{7}$$

These are equations relevant to BPS dyons and for $\beta = 0$ reduce to the Bogomol'nyi equations for uncharged monopoles. Actually all dyon solutions to (7), denoted as $(\bar\phi^a(\mathbf{r};\beta), \bar A_i^a(\mathbf{r};\beta), \bar A_0^a(\mathbf{r};\beta))$, can be obtained from the static monopole solutions $(\bar\phi^a(\mathbf{r};\beta=0), \bar A_i^a(\mathbf{r};\beta=0))$. This is achieved by the simple substitution[13]

$$\begin{aligned}
&\bar\phi_a(\mathbf{r};\beta) = \bar\phi_a(\mathbf{r}\cos\beta;0),\\
&\bar A_i^a(\mathbf{r};\beta) = \cos\beta \bar A_i^a(\mathbf{r}\cos\beta;0)],\\
&\bar A_0^a(\mathbf{r};\beta) = \mp\sin\beta\bar\phi^a(\mathbf{r}\cos\beta;0).
\end{aligned} \tag{8}$$

The $n = \pm 1$ solutions to (7) with $\beta = 0$ are well-known[3]:

$$\begin{aligned}
&\bar A_a^i(\mathbf{r};0) = \epsilon_{aij}\frac{\hat r_j}{er}\Big(1 - \frac{m_v r}{\sinh m_v r}\Big),\\
&\bar\phi_a(\mathbf{r};0) = \pm\hat r_a f\Big(\coth m_v r - \frac{1}{m_v r}\Big).
\end{aligned} \tag{9}$$

These describe BPS one-(anti-)monopole solution, centered at the spatial origin, with $g = \mp 4\pi/e$ and mass $M = g_s f = 4\pi f/e$. If the substitution (8) is made with these solutions, the results are the (classical) BPS dyon solutions with $g = \mp 4\pi/e$, $q = \mp 4\pi\tan\beta/e$ and mass $M = g_s f = 4\pi f/(e\cos\beta)$. Being a Bogomol'nyi system, there are also static multi-monopole solutions satisfying (7). But, physically, they may be viewed as representing configurations involving several of the fundamental $n = \pm 1$ monopoles described above. The latter interpretation is supported by the observation that the dimension of the moduli space of solutions with $g = \mp 4\pi n/e$ is $4n$ [12]; this is precisely the number one would expect for configurations of n monopoles, each of which is specified by three position coordinates and a U(1) phase angle associated with dyonic excitations.

The basic idea of our approach can be captured by considering the low-energy effective theory of massive vector particles in the BPS limit of SU(2) Yang-Mills-Higgs model. In the unitary gauge with the Higgs fields aligned as $\phi^a(x) = \delta_{a3}(f + \varphi(x))$, the latter model is described by the Lagrange density

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}|(\mathcal{D}^\mu W^\nu - \mathcal{D}^\nu W^\mu)|^2 - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - e^2(f+\varphi)^2 W^{\mu\dagger}W_\mu\\
&+ieF^{\mu\nu}W_\mu{}^\dagger W_\nu + \frac{e^2}{4}(W_\mu{}^\dagger W_\nu - W_\nu{}^\dagger W_\mu)(W^{\mu\dagger}W^\nu - W^{\nu\dagger}W^\mu)
\end{aligned} \tag{10}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength, and $D_\mu W_\nu$ $(\equiv \partial_\mu W_\nu + ieA_\mu W_\nu)$ the covariant derivative of charged vector field. The Higgs scalar φ, which is massless in the BPS limit, plays the role of dilaton. When the energy transfer ΔE is much smaller than the W-boson mass $m_v = ef$, the above theory may be substituted by an effective theory with the action S_{eff}, whose dynamical variables consist of the positions $\mathbf{X}_n(t)$ of W-bosons and two massless fields A_μ and φ. Ignoring contact interactions of 'heavy' W-fields and also relatively short-ranged magnetic moment interaction from (10), this low-energy action S_{eff} is easily identified, *viz.*,

$$S_{\text{eff}} = \int d^4x\{\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A\mu) - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi\} + \int dt L_{\text{eff}} \tag{11}$$

with $L_{\rm eff}$ given by

$$L_{\rm eff}=\sum_{n=1}^{N}\left\{-(m_v+g_s\varphi(\mathbf{X}_n,t))\sqrt{1-\dot{\mathbf{X}}_n^2}-q_n[A^0(\mathbf{X}_n,t)-\dot{\mathbf{X}}_n(t)\cdot\mathbf{A}(\mathbf{X}_n,t)]\right\}, \tag{12}$$

where $q_n = \pm e$ and $g_s = \frac{m_v}{f} = e > 0$, denoting the electric and dilaton charges of the W-particle, respectively. While we are eventually interested in the low energy dynamics, it is also usuful to keep the full relativistic kinetic terms for particles and solitons.

From (12) we see that the low-energy dynamics of W-particles are governed by the force law (here, $\mathbf{V}_n \equiv \frac{d}{dt}\mathbf{X}_n$)

$$\frac{d}{dt}\left[\{m_v+g_s\varphi(\mathbf{X}_n,t)\}\frac{\mathbf{V}_n}{\sqrt{1-\mathbf{V}_n^2}}\right]=q_n\mathbf{E}(\mathbf{X}_n,t)+q_n\mathbf{V}_n\times\mathbf{B}(\mathbf{X}_n,t)+g_s\mathbf{H}(\mathbf{X}_n,t)\sqrt{1-\mathbf{V}_n^2}, \tag{13}$$

where we have introduced the Higgs field strength $\mathbf{H}(x) \equiv -\nabla\varphi(x)$ together with the electric and magnetic fields $(\mathbf{E},\mathbf{B})$.

The above effective theory may also be used to derive the effective Lagrangian for a system of slowly moving W-particles. This effective particle lagrangian results once we eliminate massless fields $A_\mu(x)$ and $\varphi(x)$ from the above effective Lagrangian by using their field equations in the near-zone approximation. Assuming nonrelativistic kinematics for W-particles, we then find the slow-motion Lagrangian of the form

$$L=\frac{1}{2}\sum_{n,m}g_{ij}^{(nm)}(\mathbf{X})\dot{\mathbf{X}}_n^i\dot{\mathbf{X}}_m^j+\sum_{n>m}\frac{g_s^2-q_nq_m}{4\pi|\mathbf{X}_n-\mathbf{X}_m|} \tag{14}$$

with the inertia metric

$$\begin{aligned} g_{ij}^{(nm)}(\mathbf{X}) &= m_v\delta_{nm}\delta_{ij}-\frac{g_s^2}{4\pi}\left[\delta_{nm}\left(\sum_{k(\neq n)}\frac{1}{|\mathbf{X}_k-\mathbf{X}_n|}\right)-\frac{1-\delta_{nm}}{|\mathbf{X}_n-\mathbf{X}_m|}\right]\delta_{ij} \\ &+ \frac{q_nq_m-g_s^2}{8\pi|\mathbf{X}_n-\mathbf{X}_m|}\left[\delta_{ij}+\frac{(X_n^i-X_m^i)(X_n^j-X_m^j)}{|\mathbf{X}_n-\mathbf{X}_m|^2}\right](1-\delta_{nm}). \end{aligned} \tag{15}$$

One may discuss, for instance, low-energy scattering of two W-particles on the basis of this effective Lagrangian.

We now turn to the study of low-energy dynamics involving BPS dyons, as dictated by the time-dependent field equations of the Yang-Mills-Higgs system. Particularly important processes are those in which a single BPS dyon interacts with electromagnetic and Higgs fields—they give most direct information on the nature of effective interaction vertices involving these freedoms. Some of these processes were previously analyzed by two of us[9, 10], and here we shall recall the results obtained there.

The first case concerns an accelerating BPS dyon in the presence of a weak, uniform, electromagnetic field asymptotically[10], *viz.*, under the condition that

$$r\to\infty:\quad \frac{\phi^a}{|\phi|}B_i^a\to(\mathbf{B}_0)_i,\quad \frac{\phi^a}{|\phi|}E_i^a\to(\mathbf{E}_0)_i,\quad -\frac{\phi^a}{|\phi|}\partial_i\phi^a\to(\mathbf{H}_0)_i, \tag{16}$$

This generalizes the problem originally considered by Manton [13] some time ago. Due to the uniform asymptotic fields present, the center of dyon is expected to undergo a constant acceleration, namely, $\mathbf{X}(t)=\frac{1}{2}\mathbf{a}t^2$ (the acceleration $\mathbf{a}$ to be fixed posteriorly) in the reference frame with respect to which the dyon has zero velocity at $t=0$. To find the appropriate solution to the field equations (4) and (5), the following ansatz has been chosen in Ref. [10]:

$$\begin{aligned} &\phi^a(\mathbf{r},t)=\tilde{\phi}^a(\mathbf{r}';\beta), \\ &A_i^a(\mathbf{r},t)=-ta_i\bar{A}_0^a(\mathbf{r}';\beta)+\tilde{A}_i^a(\mathbf{r}';\beta), \\ &A_0^a(\mathbf{r},t)=-ta_i\bar{A}_i^a(\mathbf{r}';\beta)+\tilde{A}_0^a(\mathbf{r}';\beta) \end{aligned} \tag{17}$$

with

$$\tilde{\phi}^a(\mathbf{r}';\beta) = \bar{\phi}^a(\mathbf{r}';\beta) + \Pi^a(\mathbf{r}';\beta), \quad \tilde{A}_i^a(\mathbf{r}';\beta) = \bar{A}_i^a(\mathbf{r}';\beta) + \alpha_i^a(\mathbf{r}';\beta),$$
$$\tilde{A}_0^a(\mathbf{r}';\beta) = \mp \sin\beta\, \bar{\phi}^a(\mathbf{r}';\beta) + \alpha_0^a(\mathbf{r}';\beta), \tag{18}$$

where $\mathbf{r}' \equiv \mathbf{r} - \mathbf{X}(t)$, the functions $(\bar{\phi}^a(\mathbf{r};\beta),\ \bar{A}_\mu^a(\mathbf{r};\beta))$ represent the static dyon solution given by (8) (with $g = \mp 4\pi/e$ and $q = g\tan\beta$), and the yet-to-be-determined functions (Π^a,α_μ^a) are assumed to be $O(a)$ (or $O(B_0)$ or $O(E_0)$). Terms beyond $O(a)$ are ignored. Note that the functions (Π^a,α_μ^a) will account for the long-range electromagnetic and Higgs fields as well as the field deformations near the dyon core.

It then follows that the field equations (4) are fulfilled if the functions $(\Pi^a,\ \alpha_\mu^a)$ satisfy the equations

$$\tilde{B}_i^a = \mp(\tilde{D}_i + a_i)^{ab}(\cos\beta\tilde{\phi}^b \pm \tan\beta\alpha_0^b), \tag{19}$$
$$(\bar{D}_i\bar{D}_i\alpha_0)^a = -e^2\cos^2\beta\epsilon_{abc}\epsilon_{bdf}\bar{\phi}^c\bar{\phi}^f\alpha_0^d, \tag{20}$$

where $\tilde{D}_i^{ab} \equiv (D_i^{ab})_{A^a\to\tilde{A}^a}$, $\tilde{G}_c^{ji} \equiv (G_c^{ji})_{A^a\to\tilde{A}^a}$, and the suppressed dependent variable is $\mathbf{r}'$.

From these equations and the condition (16), one finds that the acceleration $\mathbf{a}$ should have the value given by

$$M\mathbf{a} = g\mathbf{B}_0 + q\mathbf{E}_0 + g_s\mathbf{H}_0, \quad (M = \frac{4\pi f}{e\cos\beta}). \tag{21}$$

Note that (21) is the equation of motion in the dyon's instantaneous rest frame, and the corresponding covariant generalization

$$\frac{d}{dt}\left(\frac{(M - g_sX_\mu H^\mu)\mathbf{V}}{\sqrt{1-\mathbf{V}^2}}\right) = g(\mathbf{B}_0 - \mathbf{V}\times\mathbf{E}_0) + q(\mathbf{E}_0 + \mathbf{V}\times\mathbf{B}_0) + g_s\mathbf{H}_0\sqrt{1-\mathbf{V}^2}. \tag{22}$$

can also be secured by further considering the implication as the Lorentz boost of our ansatz (17) is performed.

We are now ready to write down the action, which incorporates all of our findings on low-energy processes involving BPS dyons. Noting that the results of our analysis for the dyons differ from those for W-particles only by the presence of the electromagnetic duality symmetry, the desired low-energy action is given by the form

$$\begin{aligned} S_{\rm eff} &= \int d^4x\left\{\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi\right\} \\ &+ \int dt\sum_{n=1}^{N}\Big\{-(M_n + (g_s)_n\varphi(\mathbf{X}_n,t))\sqrt{1-\dot{\mathbf{X}}_n^2} \\ &- q_n[A^0(\mathbf{X}_n,t) - \dot{\mathbf{X}}_n\cdot\mathbf{A}(\mathbf{X}_n,t)] - g_n[C^0(\mathbf{X}_n,t) - \dot{\mathbf{X}}_n\cdot\mathbf{C}(\mathbf{X}_n,t)]\Big\}, \end{aligned} \tag{23}$$

where $C^\mu = (C^0,\mathbf{C})$, as a function of $F^{\mu\nu}$, are defined by

$$C^\mu(x) = -\int d^4x'(n\cdot\partial)^{-1}(x,x')n_\nu\, {}^*F^{\mu\nu}(x') + \partial^\mu\Lambda_g(x). \tag{24}$$

As one can easily verify, the above action is still invariant under the scale transformation. The desired effective Lagrangian will result if the fields $A^\mu(x)$ and $\varphi(x)$ are eliminated from the action (23) by using the above effective solutions:

$$\begin{aligned} L &= -\sum_n M_n + \frac{1}{2}\sum_n M_n\dot{\mathbf{X}}_n^2 - \frac{1}{16\pi}\sum_{n,m(\neq n)}(g_s)_n(g_s)_m\frac{|\dot{\mathbf{X}}_n - \dot{\mathbf{X}}_m|^2}{|\mathbf{X}_n - \mathbf{X}_m|} \\ &+ \frac{1}{2}\sum_{n,m(\neq n)}(q_ng_m - g_nq_m)(\dot{\mathbf{X}}_n - \dot{\mathbf{X}}_m)\cdot\omega(\mathbf{X}_n,\mathbf{X}_m) \\ &- \frac{1}{16\pi}\sum_{n,m(\neq n)}((g_s)_n(g_s)_m - q_nq_m - g_ng_m)\left\{\frac{\dot{\mathbf{X}}_n\cdot\dot{\mathbf{X}}_m}{|\mathbf{X}_n - \mathbf{X}_m|} + \frac{(\mathbf{X}_n - \mathbf{X}_m)\cdot\dot{\mathbf{X}}_n(\mathbf{X}_n - \mathbf{X}_m)\cdot\dot{\mathbf{X}}_m}{|\mathbf{X}_n - \mathbf{X}_m|^3}\right\} \\ &+ \frac{1}{8\pi}\sum_{n,m(\neq n)}\frac{(g_s)_n(g_s)_m - q_nq_m - g_ng_m}{|\mathbf{X}_n - \mathbf{X}_m|}\ . \end{aligned} \tag{25}$$

Some comments are in order as regards the slow-motion effective Lagrangian derived above. If the given system consists of BPS dyons with the same values of charges only (*i.e.*, $q_n = q$, $g_n = g$ and $(g_s)_n = \sqrt{g^2+q^2}$ for all n), all the terms in (25) which are not quadratic in velocities cancel. This is the case in which *static* multi-monopole solutions are possible, and for some given initial velocities the dynamics is governed solely by the kinetic Lagrangian of the same form as found for slowly-moving equal-charge W-particles.

Our approach, while being consistent with the moduli-space dynamics of Manton, can describe low-energy interaction of oppositely-charged BPS dyon and also process involving radiation of various massless quanta explicitly. Our discussion was entirely at the classical level, but, for an appropriately supersymmetrized system, our effective theory might be generalized to have a quantum significance. The electromagnetic duality and (spontaneously broken) scale invariance, which are manifest in our approach, may play a useful role in such endeavor. It would also be desirable to make some contact with the results of Seiberg and Witten[2]. There are some interesting related problems which require further study. Our effective action is correct when all monopoles are separated in large distance compare with the core size. If two identical monopoles overlap, the individual coordinates are not meaningful any more. We can describe the low energy dynamics by the geodesic motion on the Atiyah-Hitchin moduli space. But radiation, however weak it may be, should come out from this motion in the moduli space, including the exchange of the relative charge between two identical monopoles. Our point particle approximation does not capture this physics. It would be interesting to couple the full moduli space dynamics to the weak radiation. The present effective field theory approach should be generalized to the case of full, $N = 2$ or $N = 4$, super-Yang-Mills system. Especially the spin effect including the electric and magnetic dipole moments would appear. See Ref.[14] for the corresponding moduli-space description.

Finally, let us mention the recent work by the author and H. Min[16] where some interesting observation was made as regards to the radiation reaction and the finite-size effect in the dynamics of the BPS monopole and the duality of these effects against those of the W-particles.

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