Towards the third quantization of 2D gravity

to the memory of N. Sveshnikov

L. Chekhov[a)1] and V. Fock[b)2]

[a)] *Steklov Mathematical Institute,*
Gubkina 8, 117966, GSP–1, Moscow, Russia

[b)] *Institute of Theoretical and Experimental Physics,*
B. Cheremushkinskaya 25, 117259 Moscow, Russia

Abstract

We discuss the recent development in theory of quantum systems related to moduli spaces of Riemann curves. We construct the proper quantum observables (quantum geodesics) that are invariant with respect to a quantum modular group and satisfy the quantum algebra.

1 Introduction

The title of the present article, though being a little ambitious, resembles the fact that not all possibilities of quantum gravity as well as not all of the possible quantum finite-dimensional systems have been revealed yet. In particular, such systems possessing various discrete symmetry properties appear in many problems of mathematical physics. It is worth mentioning that the general principle of quantization—replacing the Poisson brackets $\{\ ,\ \}$ by the commutators $[\ ,\ ]$ of the generalized coordinates q_i and momenta p_j—remains unchanged since Dirac's times. However, the restrictions imposed by the symmetry considerations result in the necessity to consider rather involved quantum—some functions of q_i, p_j, depending as a rule on additional classical parameters—the spectral parameter, etc. Example for such a construction is a Hamiltonian reduction scheme [1] and all the variety of quantum integrable systems [2].

In general, having a classical system, one should first extract classical coordinates and momenta (for an involved systems, the proper Lagrangian is often unknown) and, then, imposing the Poisson bracket and exploring the symmetry properties, find proper invariant classical objects. The quantization procedure should also preserve the symmetry properties inherited from the classical model. Then, we expect that the quantum analogues of the invariant classical objects should emerge and should manifest the properties of quantum invariance.

A problem of constructing an appropriate quantum analogue of moduli spaces of various structures related to the higher genus Riemann surfaces drew much attention. In [3], the gauge theory on graphs were associated with the moduli space of flat connections. Also, worth mentioning is the series of papers [4, 5, 6], where the notion of quantum cohomologies was introduced and their relation to the associativity equations as well as the symmetry properties they manifest were extensively studied. All of the models above possess integrable model properties.

Amazingly, the moduli spaces of complex structures of Riemann surfaces were not adequately quantized. The main trouble that hinders the progress in this direction is the huge discrete symmetry group (the modular group) of the moduli space. So, the moduli spaces themselves are orbifolds of very involved structures already starting with genus two. A convenient way to deal with these symmetry groups is to consider a triangulation of the moduli space a la Penner and Kontsevich [7, 8]. This triangulation admits the description in terms of graphs, which we extensively use in the present paper. Besides, in classical case, the Kontsevich matrix model was constructed [8] that generate all observables in the model under investigation (2D quantum gravity with matter).

[1]E-mail: chekhov@genesis.mi.ras.ru.
[2]E-mail: fock@mpim-bonn.mpg.de.

When quantizing such system, however, the question of its physical content yet remains open. Indeed, it appears that already *classical* cohomology theory of moduli spaces of Riemann surfaces provides a description of *quantum* topological two-dimensional gravity. This, in turn, means that the quantization procedure we are going to describe here may be related to quantizing the two-dimensional space–time itself.

This paper is a shortened (almost all of the proofs are omitted) version of [9].

2 Classical moduli spaces

We begin with a classical description of Teichmüller spaces of complex structures on Riemann surfaces with holes (punctures).

Teichmüller space $\mathcal{T}^h$ is a space of complex structures on S modulo diffeomorphisms. This complex structure inambiguously determines a metric on the Riemann surface, while, in this metric, a neighborhood of each hole corresponds to the exponential mapping from the punctured disc to a half-infinite cylinder. This means that the Riemann surface becomes nonclosed in a sense of metric. This nonclsedness is, however, "weak," i.e., the total boundary length is finite and it corresponds to glueing of half-infinite cylinders of fixed finite widths to a surface with a number of handles.

According to the Poincaré uniformization theorem, any complex surface S of a constant negative curvature (equal -1 in most of the applications) is a quotient of the upper half-plane $\mathbf{H}_+$ endowed with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \tag{2.1}$$

by a discrete subgroup $\Delta(S)$ of the automorphism group $PSL(2, \mathbf{R})$,

$$S = \mathbf{H}_+/\Delta(S).$$

In metric (2.1), geodesics are either half-infinite circles with endpoints at the real line $\mathbf{R}$ or vertical half-lines; all points of the boundary $\mathbf{R}$ are at infinite distance from each other and from any interior point.

The subgroup $\Delta(S)$ is canonically isomorphic to the fundamental group of the surface $\pi_1(S)$, i.e., to the group of all classes of homotopically equivalent embeddings of the circle S^1 into the Riemann surface. Furthermore, $\Delta(S)$ is determined by the complex structure on the surface up to the conjugation by an element of $PSL(2, \mathbf{R})$. Therefore, we have an imbedding

$$\mathcal{T}^h(S) \to \mathrm{Hom}\big(\pi_1(S) \to PSL(2, \mathbf{R})\big)/PSL(2, \mathbf{R})$$

with the property that the image of any loop is a matrix with two distinct real eigenvalues.

Any homotopy class of closed curves contains a unique *closed geodesic*, while the homotopy classes of closed curves on a surface are in the one-to-one correspondence with the conjugacy classes of its fundamental group Δ.

Let γ be an element of $\pi_1(S)$ and $l(\gamma)$ the length of the corresponding geodesics. Then

$$l(\gamma) = \left|\log \frac{\lambda_1}{\lambda_2}\right|,$$

where λ_1 and λ_2 are the (different) eigenvalues of the element of $PSL(2, \mathbf{R})$ that corresponds to γ.

2.1 Teichmüller space of surfaces with holes $\mathcal{T}^H(S)$. The graph technique

Here we recall a description of the moduli space $\mathcal{T}^H(S)$ in terms of oriented graphs.

Definition 1 A *fat graph* (oriented graph) Γ is a graph with the given cyclic ordering of edges entering each (usually, three-valent) vertex.

To each (unoriented) edge we put into the correspondence the real number $Z \in \mathbf{R}$; the set of all such numbers is $\{Z_\alpha | \alpha \in E(\Gamma)\}$ where $E(\Gamma)$ is the set of all (unoriented) edges of the graph Γ. Analogously, we denote by $E(\vec{\Gamma})$ the set of all oriented edges $\vec{\alpha}$ of the graph Γ.

Consider the set of all the oriented graphs of a given genus g and number of faces (punctures) n. Then the following assertion is true.

Theorem 1 (Strebel [10]). *There is a one-to-one correspondence between the set of points of $\mathcal{T}^H(S)$ and the set of oriented graphs Γ of the given genus g and number of punctures n supplied with positive numbers (lengths) of the corresponding edges.*

This construction provides us with the cell decomposition of the corresponding moduli space.

In what follows, we neglect all the lower dimensional cells (which correspond to graphs with higher-order vertices) in this decomposition and deal with the three-valent graphs only.

2.1.1 Constructing the Fuchsian group

We propose the explicit way to construct the Fuchsian group $\Delta(S) \subset PSL(2, \mathbf{R})$ such that $S = \mathbf{H}_+/\Delta(S)$.

We modify the original graph Γ at each vertex by disconnecting edges at the vertex and connecting them by three "short" edges forming a triangle (Fig. 1). The edges in all of the triangles are counter-clockwise oriented.

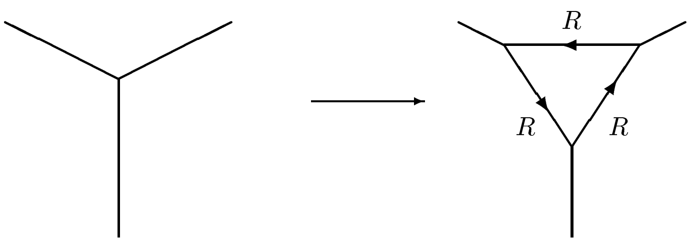


Fig. 1

To each "old" (unoriented) "long" edge α we associate the matrix of the Möbius transform

$$X_{z_\alpha} = \begin{pmatrix} 0 & -\mathrm{e}^{Z_\alpha/2} \\ \mathrm{e}^{-Z_\alpha/2} & 0 \end{pmatrix}, \tag{2.2}$$

and to each "short" oriented edge we associate the transition matrix of the "right turn"

$$R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.3}$$

The composition of two matrix R is the matrix of the "left turn"

$$L \equiv R^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad LR = RL = X_z^2 = -I. \tag{2.4}$$

We denote by $\widetilde{\Gamma}$ the resulting graph with a number of "short" oriented edges inserted in places of the oriented vertices of the initial graph Γ.

Remark 1 Let *ideal triangle* be any triangle in $\mathbf{H}_+$ with vertices lying on the boundary (infinitely distant from any point in the inner region). Consider two ideal triangles with vertices at the ponts $-1, 0, \infty$ and $\mathrm{e}^{z}, \infty, 0$, respectively. Then the Möbius transforms (2.3) and (2.4) permute the vertices of the first triangle, while transform (2.2) maps one triangle to another.

Let us see explicitly how one may construct the element of the group $\Delta(S)$. Consider S to be a torus with one hole, which is composed from two ideal triangles "1" and "0" depicted in Fig. 2. (Opposite sides of the resulting ideal qudrangle are to be identified.) Then, one of the elements of the group Δ should identify two geodesics: $(\mathrm{e}^{Z},\infty) \to (0,-1)$ (another generating hyperbolic element should transform $(-1,\infty)$ to $(0,\mathrm{e}^{Z})$). Recall that each element of $PSL(2,\mathbf{R})$ is inambiguously detrmined if we know images of three distinct points on the real line. This means that in order to determine the element in hands, we need to know the image of one more point, say the zero point. This image lies between the points -1 and 0, so we denote it by $-1/(1+\mathrm{e}^{Y})$. The points $\{-1,-1/(1+\mathrm{e}^{Y}),0\}$ determine the ideal triangle "2." Now we have to construct the element of $PSL(2,\mathbf{R})$ that transform triangle "1" into triangle "2." This can be done in four steps using operators (2.2)–(2.4):

$$\begin{aligned} X_z(0,\mathrm{e}^{Z},\infty) &= (\infty,-1,0) \\ R(\infty,-1,0) &= (-1,0,\infty) \\ X_y(-1,0,\infty) &= (\mathrm{e}^{Y},\infty,0) \\ L(\mathrm{e}^{Y},\infty,0) &= \left(\frac{-1}{1+\mathrm{e}^{Y}},0,-1\right). \end{aligned} \tag{2.5}$$

Therefore, the desired element is $LX_yRX_z \in PSL(2,\mathbf{R})$. It is easy to verify (see Definition 2 and Example 2 below) that this element corresponds to a *closed path* in the graph describing the torus with one hole.

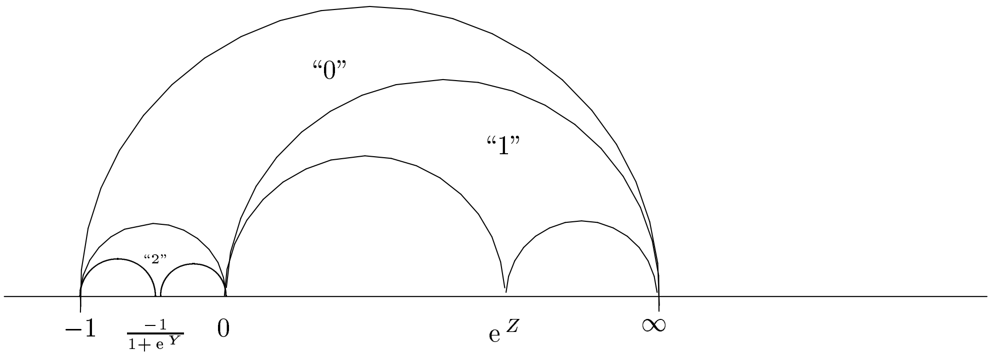


Fig. 2

Let us call the "operators of the right and left turns," R_z and L_z, the products

$$R_z \equiv RX_z = \begin{pmatrix} \mathrm{e}^{-Z/2} & -\mathrm{e}^{Z/2} \\ 0 & \mathrm{e}^{Z/2} \end{pmatrix}, \tag{2.6}$$

$$L_z \equiv LX_z = \begin{pmatrix} \mathrm{e}^{-Z/2} & 0 \\ -\mathrm{e}^{-Z/2} & \mathrm{e}^{Z/2} \end{pmatrix}. \tag{2.7}$$

We introduce now the notion of *geodesic* on the graph. For this, we need two more definitions.

Definition 2 Let a *closed path* in the graph Γ (or, $\widetilde{\Gamma}$) is any oriented path, which starts and terminates at some oriented edge (point) of the corresponding graph. To each such path we put into the correspondence the product of matrices $P_{z_1\cdots z_n} = L_{z_n}L_{z_{n-1}}R_{z_{n-2}}\cdots R_{z_2}L_{z_1}$, where the matrices L_{z_i} or R_{z_i} are on the ith place depending on which turn—left or right—the path is going on the ith step.

Note that, in principle, any edge $\vec{\alpha}_z$ may enter arbitrary many times in $P_{z_1\cdots z_n}$; however, we have to distinguish between the original path and their repetitions $P_{z_1\cdots z_{2n}} = P_{z_1\cdots z_n}P_{z_1\cdots z_n}$, i.e., P^3, P^4, etc. This distinction will be important in the quantum case.

Proposition 1 [11] 1. *There is a one-to-one correspondence between the set of all primitive oriented paths $\{P_{z_1\cdots z_n}\}$ and closed geodesics $\{\gamma\}$ on the moduli space. Moreover, the length $L(\gamma)$ of a geodesic can be determined as follows:*

$$G(\gamma) \equiv 2\cosh L(\gamma)/2 = \operatorname{tr} P_{z_1\cdots z_n}. \tag{2.8}$$

2. *Considering the universal covering T_Γ of the graph Γ, $\Gamma = T_\Gamma/\Delta_\Gamma$, where Δ_Γ is the Fuchsian group of the graph Γ, we obtain that there is a one-to-one correspondence between the set of all closed geodesics and the primitive conjugacy classes $\{\sigma_i\} \subset \Delta_\Gamma$.*

Summing up, we have the following chain of the one-to-one correspondences:

$$\left\{\begin{array}{c}\text{conjugate}\\ \text{classes of}\\ \Delta(S)\subset PSL(2,\mathbf{R})\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{closed}\\ \text{geodesics}\\ \text{on } S\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{closed}\\ \text{paths}\\ \text{in (any of)}\\ \text{the graph } \Gamma(S)\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{elements}\\ \text{of homotopy}\\ \text{group } \pi_1(S)\end{array}\right\}.$$

Remark 2 Note that both matrices (2.6) and (2.7) have the form $\left\{\begin{array}{cc}(+) & (-)\\ (-) & (+)\end{array}\right\}$, so *any product* of such matrices will be of the same form; meanwhile, in a diagonal term of any such product, *all* summands enter with the plus sign and, in an antidiagonal term, with the minus sign.

2.2 Weil–Peterson forms

A canonical Poisson structure called the Weil–Peterson structure exists on the set of coordinates $\{Z_\alpha\}$. This structure is degenerate, and the Casimir functions are the lengths of geodesics surrounding holes. Fixing all these lengths, we obtain a symlpectic leaf. On every such leaf, the Poisson structure can be inverted as to give a symplectic structure. The resulting degenerate two-form is also called the *Weil–Peterson form.*

In terms of the constructed coordinates, these forms read as follows [7]:

$$w_{WP} = \sum_{\alpha\in E(\vec{\Gamma})} dZ_\alpha \wedge dZ_{\alpha(1)} \tag{2.9}$$

and

$$P_{WP} = \sum_{\alpha\in E(\vec{\Gamma})} \frac{\partial}{\partial Z_\alpha} \wedge \frac{\partial}{\partial Z_{\alpha(1)}}, \tag{2.10}$$

where $\alpha(1)$ is the oriented edge obtained from α by rotating by the angle $2\pi/3$ clockwise around the vertex these edges are starting with.

Denote α^{-1} the oriented edge that are opposite to an edge α. Then, the Weil–Peterson Poisson brackets read

$$\{Z_\alpha, Z_\beta\} = \left\{\begin{array}{rll} 1 & \text{if} & \alpha^{\pm1}(1) = \beta \text{ or } \beta^{-1} \quad\quad \text{and } \alpha^{\mp1} \neq \beta, \beta(1),\\ -1 & \text{if} & \alpha^{\pm1} = \beta(1) \text{ or } \beta^{-1}(1) \quad \text{and } \alpha^{\mp1} \neq \beta, \beta(1),\\ 2 & \text{if} & \alpha^{\pm1} = \beta^{\pm1}(1) \text{ or } \beta^{\mp1}(1),\\ -2 & \text{if} & \alpha^{\pm1}(1) = \beta^{\pm1} \text{ or } \beta^{\mp1},\\ 0 & & \text{otherwise.}\end{array}\right. \tag{2.11}$$

2.3 Classical projective (modular) transformations

In [11], the projective transformations on graphs that preserve the Poisson structure (2.11) were obtained. For this, let us note that there exists a natural operation called the *flip* or Whitehead move, which corresponds to transitions between graphs. We must determine such a transformation of the variables Z_α that, first, preserve the Poisson structure and, second, after a series of flips that transform a graph to itself, the resulting transformation should be the identity. This implies the presence of another relation called the *pentagon identity.* It appears that the transformations depicted in Fig. 3 (2.12) satisfy both the demands.

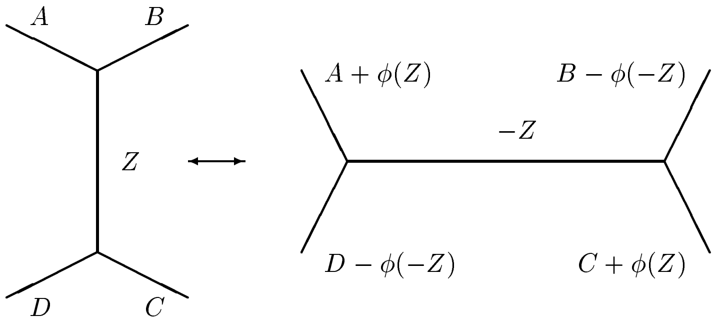


Fig. 3

In the classical case [11],

$$\begin{aligned}\{A,B,C,D,Z\} &\to \{A+\phi(Z), B-\phi(-Z), C+\phi(Z), D-\phi(-Z), -Z\}\\ &\equiv \{\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D},\widetilde{Z}\}, \\ \phi(Z) &= \log(\mathrm{e}^{Z}+1).\end{aligned} \tag{2.12}$$

Lemma 1 *Transformation* (2.12) *preserves the products over paths, so the classical geodesic length is a modular invariant function.*

Example 1 Consider the geodesic γ_j surrounding the jth puncture (around-the-face geodesic). Then, obviously, it has a form $\mathrm{tr}\, R_{Z_1}R_{Z_2}\dots R_{Z_n}$, or $\mathrm{tr}\, L_{Z_1}L_{Z_2}\dots L_{Z_n}$ depending on the orientation. One can easily see that, in this case, all of the matrices R_x (L_x) are upper (lower) triangular and formula (2.8) implies that

$$l_{\gamma_j} = \left|\sum_{i=1}^{n} Z_i\right|.$$

2.4 Poisson algebra of geodesics

The functions $\{G_\gamma\}$ (2.8) were studied by several authors and mainly by W. Goldman [12]. Among their nice properties let us mention here that they generate a Poisson algebra (w.r.t. the multiplication and the Weil-Petersson Poisson bracket) over **Z**. (It means that a product and a Poisson bracket of two such functions is a linear combination of such functions with integer coefficients.)

In this section, we consider the Poisson structure of geodesics. Two nonintersecting geodesics have trivial bracket and, due to the linearity property, we may consider only "simple" intersections of two geodesics G^1 and G^2 of the form

$$G^1 = \mathrm{tr}^{\,1} \dots X^1_C R^1 X^1_Z L^1 X^1_A \dots, \tag{2.13}$$

$$G^2 = \mathrm{tr}^{\,2} \dots X^2_B L^2 X^2_Z R^2 X^2_D \dots, \tag{2.14}$$

respectively (cf. Figs. 3 and 6 for the quantum case). (Both these geodesics describe nontrivial laminations of the surface.) The superscripts 1 and 2 pertain to the matrix spaces over which the traces are taken.

If we consider two geodesics corresponding to trivial laminations, e.g., $\dots X^1_A R^1 X^1_Z R^1 X^1_B \dots$ and $\dots X^1_C R^1 X^1_Z R^1 X^1_D \dots$, then, after the flip of the edge Z, the resulting geodesics $\widetilde{G}^1$ and $\widetilde{G}^2$ do not intersect and, therefore, have trivial bracket. So, by virtue of Lemma 1, the initial bracket is trivial as well.

First, we need a bracket between X^1_C and X^2_B, which has a simple r-matrix structure,

$$\{X^1_C, X^2_B\} = \frac{1}{4}(\sigma^1_3 \otimes \sigma^2_3) X^1_C \otimes X^2_B, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.15}$$

Then, direct calculations give

$$\begin{aligned}\{G_1, G_2\} = {} & \dots \frac{1}{4}(X^1_C R^1)\otimes(X^2_B L^2)\bigl\{2\sigma^1_3 X^1_Z \otimes \sigma^2_3 X^2_Z - 4(e^1_{22}\otimes e^2_{11} + e^1_{11}\otimes e^2_{22})\bigr\} \times \\ & \times (L^1 X^1_A)\otimes(R^2 X^2_D)\dots .\end{aligned} \tag{2.16}$$

Hereafter, we use the standard notation for the matrix e_{ij} to be the matrix with the unity at the ith line and jth column and zero otherwise.

Eventually we obtain that

$$\{G_1, G_2\} = \frac{1}{2}(\operatorname{tr} G_{\mathrm{H}} - \operatorname{tr} G_{\mathrm{I}}), \tag{2.17}$$

where G_{I} is the geodesic that is obtained by erasing the edge Z and joining together the edges "A" and "D" as well as "B" and "C" in a natural way; the geodesic G_{H} passes over the edge Z twice, so it has the form $\operatorname{tr}\ \ldots X_C R_Z R_D \ldots X_B L_Z L_A \ldots$. This is nothing but the relations obtained in [12] in the continuous parameterization (the classical Turaev–Viro algebra).

Remark 3 By the polylinearity property of the Poisson bracket, relation (2.17) can be naturally generalized to the case where there are more than one intersection point, or there are multiple intersections at the same point.

Example 2 Consider the torus with three generators X, Y, Z and the Poisson algebra

$$\{X, Y\} = \{Y, Z\} = \{Z, X\} = 2$$

(the Casimir element is $X + Y + Z$). Then, the three simple geodesics are (cf. (2.5))

$$\begin{aligned} G_X &= \operatorname{tr} L_Y R_Z = e^{-Y/2-Z/2} + e^{-Y/2+Z/2} + e^{Y/2+Z/2}, \\ G_Y &= \operatorname{tr} L_Z R_X = e^{-Z/2-X/2} + e^{-Z/2+X/2} + e^{Z/2+X/2}, \\ G_Z &= \operatorname{tr} L_X R_Y = e^{-X/2-Y/2} + e^{-X/2+Y/2} + e^{X/2+Y/2}. \end{aligned} \tag{2.18}$$

Introducing the geodesic $\widetilde{G_Z}$ obtained from G_Z by the flip transformation,

$$\widetilde{G_Z} = \operatorname{tr} R_Z R_X L_Z L_Y = e^{-X/2-Y/2-Z} + e^{X/2-Y/2}(e^{-Z} + e^{Z} + 2) + e^{X/2+Y/2+Z},$$

we obtain

$$\{G_X, G_Y\} = \frac{1}{2}\widetilde{G_Z} - \frac{1}{2}G_Z.$$

3 Quantization

Once the graph Γ (or, in another words, the decomposition of the surface S into triangles) is chosen, the corresponding Teichmüller space $\mathcal{T}^H(S)$ can be easily quantized. Consider the $*$-algebra $\mathcal{T}^\hbar(\Gamma)$ generated by real generators $\{Z_\alpha^\hbar | \alpha \in E(\Gamma)\}$, $(Z_\alpha^\hbar)^* = Z_\alpha^\hbar$ with the defining relations

$$[Z_\alpha^\hbar, Z_\beta^\hbar] = 2\pi i\hbar\{Z_\alpha, Z_\beta\} \qquad \text{cf. (2.11).} \tag{3.1}$$

This algebra has an obvious center generated by

$$\left\{ P_\gamma | \gamma \in F(\Gamma),\ P_\gamma = \sum_{\alpha\in\gamma} Z_\alpha^\hbar \right\}. \tag{3.2}$$

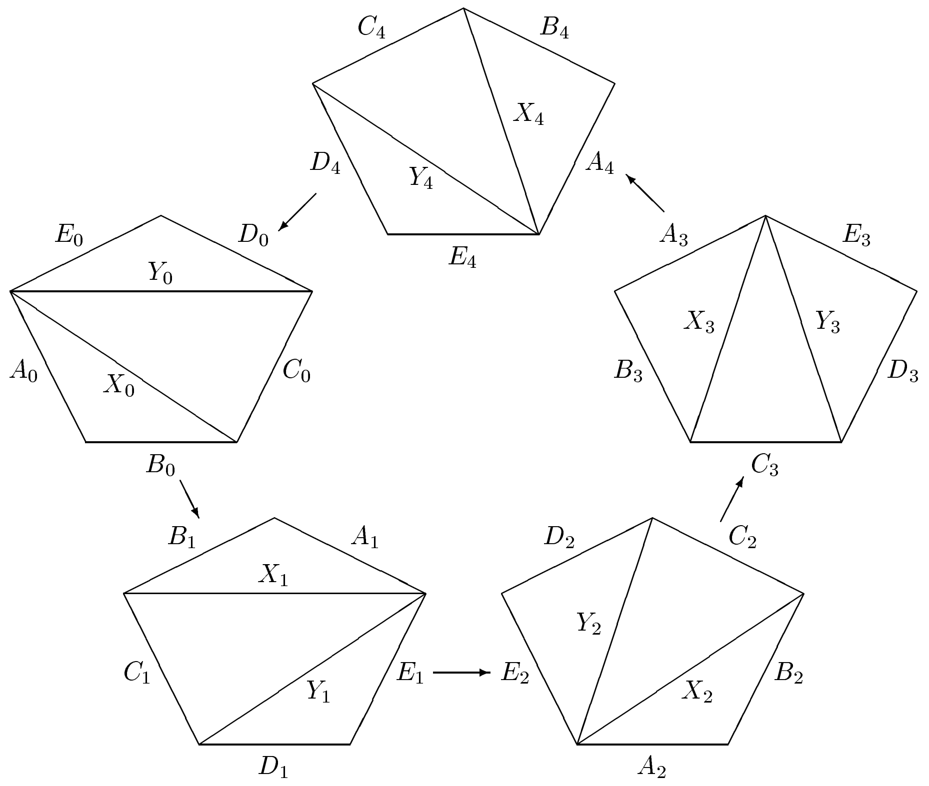
One can easily describe all irreducible $*$-representations of this algebra using the Stone–von Neumann theorem. An irreducible representaion is unambiguously fixed by the values of the operators P_γ for all $\gamma \in F(\Gamma)$ which must be scalars. For example one can represent all operators $Z_\alpha^\hbar$ in $L^2(\mathbf{R}^n)$, where $n = \frac{1}{2}(\sharp E(\gamma) - \sharp F(\Gamma))$, by linear combinations with rational coefficients standing by constants and the operators x_i and $i\frac{\partial}{\partial x_i}$, where $\{x_i | i = 1, \ldots, n\}$ is a standard coordinate system on $\mathbf{R}^n$. Now our task is to identify the $*$-algebras constructed using different graphs Γ and Γ' corresponding to a given surface S . In order to make this identification we just construct a $*$-homomorphism $K(\Gamma, \Gamma') : \mathcal{T}^\hbar(\Gamma) \to \mathcal{T}^\hbar(\Gamma')$ of the $*$-algebra generated by $\{Z_{\alpha'}^\hbar | \alpha' \in E(\Gamma')\}$ to the algebra generated by $\{Z_\alpha^\hbar | \alpha \in E(\Gamma)\}$. We require this homomorphism to have the following properties:

1. *Classical limit.* We demand that the algebra homomorphism should tend to the classical homomorphim of the algebras fo function on $\mathcal{T}^H$ when the parameter $\hbar$ tends to zero.

2. *Path independence.* We demand that if we have three graphs Γ, Γ' and Γ'' then the homomorphisms should satisfy the condition $K(\Gamma'',\Gamma')K(\Gamma',\Gamma) = K(\Gamma'',\Gamma)$.

Using the latter demand, one can obviously reconstruct the homomorphism $K(\Gamma',\Gamma)$ for any Γ and Γ' once one knows these homomorphisms for pairs of graphs related by single flips. However an arbitrary set of such flip homomorphisms *a priori* does not satisfy the path independence condition since one can get one graph from another by different sequences of flips. In other words, one must check that if a sequence of flips does not change the graph, the corresponding product of algebra homomorphisms is the identical one.

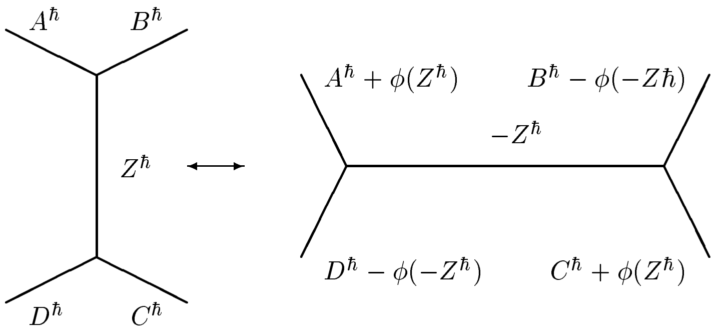
Fortunately one can check the latter condition only for one distinguished sequence of flips since others are just compositions of this one. Consider two edges having exactly one common vertex. Then, a sequence of five flips of these edges (such that we never flip the same edge twice consequtively) does not change the graph. This becomed more geometrically transparent if we consdier the dual graph where the two edges correspond to two edges separating three triangles forming a pentagon. A pentagon can be cut into three triangles in only five possible ways which are related by flips (Fig. 4)



(3.3)

Fig. 4

As for the flip homomorphism satisfying the above conditions, we first draw Fig 5. On the left-hand side of Fig. 5, a fragment of the graph Γ is shown together with the algebra elements associated to the edges. On the right-hand side of Fig. 5, the corresponding fragment of the graph Γ' together with the operators that are images of operators corresponding to the edges in the former algebra is drawn. The remainder of the graphs as well as the operators assigned to the remainder of the edges of Γ and Γ' coincide.



(3.4)

Fig. 5

The function $\phi(z)$ is a real-analytic function of z depending on $\hbar$ as on a parameter,

$$\phi(z) = -\frac{\pi\hbar}{2} \int_\Omega \frac{e^{-ipz}}{\sinh(\pi p)\sinh(\pi\hbar p)} dp, \tag{3.5}$$

and the contour Ω goes along the real axis bypassing the origin from above.

Verification of the classical limit condition is straightforward and follows from the elementary properties of the function $\phi(x, \hbar)$:

$$\lim_{\hbar\to 0} \phi(x, \hbar) = \log(e^x + 1). \tag{3.6}$$

The function (3.5) and $F(x, \hbar)$,

$$F(x, \hbar) := \exp(-\frac{1}{4} \int_\Omega \frac{e^{-ipx}}{p\sinh(\pi p)\sinh(\pi\hbar p)} dp), \tag{3.7}$$

first appeared in [13] possesses the following properties.

1. $$\lim_{x\to-\infty} \phi(x, h) = 0. \tag{3.8}$$

2. $$\lim_{h\to 0} \phi(x, \hbar) = \log(e^x + 1). \tag{3.9}$$

3. $$\frac{\phi(x + i\pi\hbar) - \phi(x - i\pi\hbar)}{2i\pi\hbar} = \frac{1}{1 + e^{-x}}. \tag{3.10}$$

$$\frac{\phi(x + i\pi) - \phi(x - i\pi)}{2i\pi} = \frac{1}{1 + e^{-\frac{1}{\hbar}x}}. \tag{3.11}$$

4. $$2\pi i\hbar \frac{\partial}{\partial x} \log F(x, \hbar) = \phi(x, \hbar). \tag{3.12}$$

5. $$F(x + i\pi\hbar, \hbar) = F(x - i\pi\hbar, \hbar)(1 + e^x). \tag{3.13}$$

$$F(x + i\pi) = F(x - i\pi)(1 + e^{\frac{1}{\hbar}x}). \tag{3.14}$$

6. $$\phi(x, \hbar) = \hbar\phi\left(\frac{1}{\hbar}x, \frac{1}{\hbar}\right). \tag{3.15}$$

Now it is straightforward to check that the following assertion is valid.

Lemma 2 *Transformations* (3.4) *satisfy the pentagon identity.*[1]

3.0.1 $(\hbar \to \frac{1}{\hbar})$-invariance.

The algebras generated by $\{Z^\hbar_\alpha\}$ corresponding to different values of $\hbar \geq 0$ are obviously isomorphic to each other. However these isomorphisms are in general not canonical. It means that *a priory* it is not possible to define an isomorphism commuting with the action of the mapping class group.

[1] This result was independently obtained by R. Kashaev [14].

It turns out that for some pairs of values of the parameter $\hbar$ such eqivariant isomorphism does exist. Indeed consider the $*$-algebra isomorphism $D(\Gamma) : \mathcal{T}^\hbar(\Gamma) \to \mathcal{T}^{\frac{1}{\hbar}}(\Gamma)$ given on generators by

$$D(\Gamma) : Z^\hbar_\alpha \mapsto \frac{1}{\hbar} Z^{\frac{1}{\hbar}}_\alpha \tag{3.16}$$

One can easily check that for two different graphs Γ and Γ' related by a flip one has

$$K(\Gamma, \Gamma')D(\Gamma) = D(\Gamma')K(\Gamma, \Gamma'). \tag{3.17}$$

what means that in fact this isomorphism does not depend on the graph we have chosen to define it. For this reason we shall denote it below as $D(S)$.

The proof of the equality (3.17) is straightforward provided one uses Property 6 above. This property shows, in particular, that representations of the mapping class groups corresponding to $\hbar$ and $\frac{1}{\hbar}$ are isomorphic. It means also that the quantization has two isomorphic classical limits $\hbar \to 0$ and $\hbar \to \infty$. If we assume that we have here the quantum Liouville theory, we can interpret the parameter $\hbar$ as a coupling constant and the $\hbar \leftrightarrow \frac{1}{\hbar}$ symmetry as a "week–strong coupling constant duality."

3.1 Geodesic length operators

The aim of this paragraph is to imbed the algebra of geodesics (2.8) into a suitable completion of the constructed algebra $\mathcal{T}^\hbar(S)$.

The function G_γ (2.8) can be expressed for any γ in terms of graph coordinates on $\mathcal{T}^H$,

$$G_\gamma \equiv \operatorname{tr} P_{Z_1 \cdots Z_n} = \sum_{j \in J} e^{\frac{1}{2} \sum_{\alpha \in E(\Gamma)} m_j(\gamma,\alpha) z_\alpha}, \tag{3.18}$$

where $m_j(\gamma, \alpha)$ are certain integer numbers and J is just a finite set of indices numerating the terms in (3.18).

Let us now define and formulate the properties of quantum analogues of these functions.

Denote by $\widehat{\mathcal{T}}^\hbar$ a completion of the algebra $\mathcal{T}^\hbar$ containing e^{xZ_α} for any real x. Let for any closed path γ on S, the operator $G^\hbar_\gamma \in \hat{\mathcal{T}}^\hbar$ be given by

$$G^\hbar_\gamma \equiv {}^\times_\times \operatorname{tr} P_{z_1 \ldots z_n} {}^\times_\times = \sum_{\substack{j \in J \\ \kappa \in \{j\}}} e^{\frac{1}{2} \sum_{\alpha \in E(\Gamma)} \left(m_j(\gamma,\alpha) Z^\hbar_\alpha + 2\pi i \hbar c^\kappa_j(\gamma,\alpha) \right)}. \tag{3.19}$$

Here some quantum ordering ${}^\times_\times \cdot {}^\times_\times$ is assumed. The main problem is to find the ordering that satisfy all of the conditions below. In (3.19), the numbers $m_j(\gamma, \alpha)$ are the same as in (3.18) and integer coefficients $c^\kappa_j(\gamma, \alpha)$ are to be determined from the procedure of the quantum ordering.

Note that the operators $\{G^{\frac{1}{\hbar}}_\gamma\}$ can be considered as belonging to the algebra $\hat{\mathcal{T}}^\hbar$ due to the isomorphism $D(S)$ given by (3.16). In terms of the generators of $\hat{\mathcal{T}}^\hbar$, they are obviously given by

$$G^{\frac{1}{\hbar}}_\gamma = \sum_{j \in J} e^{\frac{1}{2\hbar} \sum_{\alpha \in E(\Gamma)} \left(m_j(\gamma,\alpha) Z^\hbar_\alpha + 2\pi i c^\kappa_j(\gamma,\alpha) \right)}, \tag{3.20}$$

Now let us formulate the defining properties of quantum geodesics.

1. *Modular invariance.* The modular group $\Delta(S)$ (3.4) preserves the set $\{G^\hbar_\gamma\}$, i.e., for any $\delta \in \Delta(S)$ and any closed path γ, we have $\delta(G^\hbar_\gamma) = G^\hbar_{\delta\gamma}$.

2. *Geodesic algebra.* The product of two quantum geodesics is a linear combination of quantum geodesics governed by the skein relation [15].

3. *Unorientness.* Trace of inverse geodesics operator coincides with the quantum trace of the direct one.

4. *Exponents of geodesics.* Being raised to any power n, a quantum geodesics $G_\gamma = 2\cosh L(\gamma)/2$ is to be expanded into the linear combination of quantum geodesics $G_{k\gamma}$, $k = 1, \ldots, n$; the (binomial) coefficients are to be the same as in the relation

$$(2\cosh L(\gamma)/2)^n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{k}{n} 2\cosh\Big((n-2k)L(\gamma)/2\Big).$$

5. For any γ and γ', the operators $G^{\hbar}_{\gamma}$ and $G^{\frac{1}{\hbar}}_{\gamma'}$ commute.

6. If two closed paths γ and γ' do not intersect, then the operators $G^{\hbar}_{\gamma}$ and $G^{\hbar}_{\gamma'}$ commute.

Definition 3 A *simple geodesic* is a geodesic that passes through each edge of a graph no more than once.

Note that the following lemma holds true.

Lemma 3 *If the quantum transformations* (3.4) *transform a simple geodesic G into the simple geodesic $\widetilde{G}$, than, for these geodesics, the quantum ordering* $\substack{\times \\ \times}\cdot\substack{\times \\ \times}$ *is the Weyl ordering and all* $c^{\kappa}_{j}(\gamma,\alpha)\equiv 0$ *in* (3.19).

So, Lemma 3 is a partial analogue of Lemma 1. Next step is to consider modular transformations that do not preserve the simplicity property and compare the result with the one obtained from algebraic relations.

We denote the Weyl ordering by a usual normal ordering symbol $:\ldots:$, i.e.,

$$:\mathrm{e}^{a_1}\,\mathrm{e}^{a_2}\cdots\mathrm{e}^{a_n}:\equiv \mathrm{e}^{a_1+\cdots+a_n}\quad\text{for any}\quad \{a_i\}.$$

Example 3 Quantum geodesics for torus. For the torus, there are three simple geodesics, which, in the Weyl ordering, have exactly the form (2.18). Then, the geodesics $\widetilde{G}_Z$ obtained from G_Z by the flip transformation is

$$\widetilde{G}_Z = \mathrm{e}^{-X/2-Y/2-Z} + \mathrm{e}^{X/2-Y/2-Z} + \mathrm{e}^{X/2-Y/2}2\cos(\pi\hbar) + \mathrm{e}^{X/2-Y/2+Z} + \mathrm{e}^{X/2+Y/2+Z}. \tag{3.21}$$

On the other hand, explicit calculations show that

$$[G_X, G_Y] = 2\sin(\pi\hbar/2)(\widetilde{G}_Z - G_Z), \tag{3.22}$$

i.e., the elements of the algebra of geodesics coincide with the elements that are obtained from the simple geodesics by the modular transformation (3.4). Moreover, if we consider merely the *product* of two geodesics, the results will be

$$G_X G_Y = \mathrm{e}^{i\pi\hbar/2}\widetilde{G}_Z + \mathrm{e}^{-i\pi\hbar/2}G_Z, \tag{3.23}$$

so the algebra of geodesics appears to be closed w.r.t. the product as well.

The question arises which modification of the Weyl ordering should we choose to obtain the proper modular-invariant set of geodesics (3.19)? We partially answer this question in the next section.

3.2 Algebra of quantum geodesics

Let G_1 and G_2 be two simple geodesics intersecting with each other as depicted in Fig. 6.[2] So, for G_1 and G_2, the formula (3.19) implies, by virtue of Lemma 3, the mere Weyl ordering.

[2]In fact, all of the reasonings below are valid also for nonsimple geodesics such that the quantities $\xi_{(ca)}$ and $\xi_{(bd)}$ commute with $Z^{\hbar}$ and with each other.

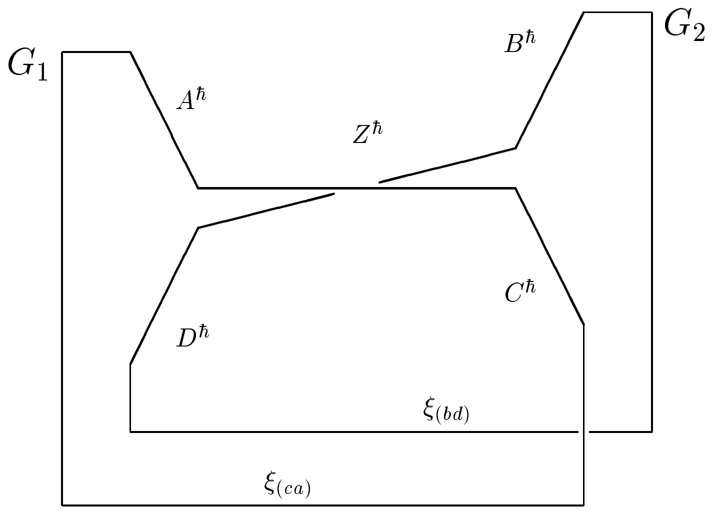


Fig. 6

Let us consider the product G_1G_2,

$$
\begin{aligned}
G_1G_2 \;&=\; :\mathrm{tr}\,_1\xi_{(ca)}X_AR_ZL_C::\mathrm{tr}\,_2\xi_{(bd)}X_DL_ZR_B: \\
&=\; :\mathrm{tr}\,_1\xi_{(ca)}\begin{pmatrix} \mathrm{e}^{\frac{A}{2}+\frac{Z}{2}-\frac{C}{2}} & -\mathrm{e}^{\frac{A}{2}+\frac{C}{2}+\frac{Z}{2}} \\ \mathrm{e}^{-\frac{Z}{2}-\frac{C}{2}-\frac{A}{2}}+\mathrm{e}^{\frac{Z}{2}-\frac{C}{2}-\frac{A}{2}} & -\mathrm{e}^{-\frac{A}{2}+\frac{Z}{2}+\frac{C}{2}} \end{pmatrix}:\times \\
&\qquad\times\,\mathrm{tr}\,_2\xi_{(bd)}\begin{pmatrix} \mathrm{e}^{\frac{D}{2}-\frac{Z}{2}-\frac{B}{2}} & -\mathrm{e}^{\frac{D}{2}+\frac{B}{2}+\frac{Z}{2}}-\mathrm{e}^{\frac{D}{2}+\frac{B}{2}-\frac{Z}{2}} \\ \mathrm{e}^{-\frac{D}{2}-\frac{Z}{2}-\frac{B}{2}} & -\mathrm{e}^{-\frac{D}{2}-\frac{Z}{2}+\frac{B}{2}} \end{pmatrix}:.
\end{aligned}
\tag{3.24}
$$

¿From now on, in all formulas the normal ordering is assumed. Let us release the quantities

$$\widehat{\xi}_{(ca)} \equiv LX_C\xi_{(ca)}X_AR \quad \text{and} \quad \widehat{\xi}_{(bd)} \equiv RX_B\xi_{(bd)}X_DL$$

Then, after some algebra we have (cf. (2.17))

$$G_1G_2 = \mathrm{e}^{-i\pi\hbar/2}G_Z + \mathrm{e}^{i\pi\hbar/2}\widetilde{G}_Z, \tag{3.25}$$

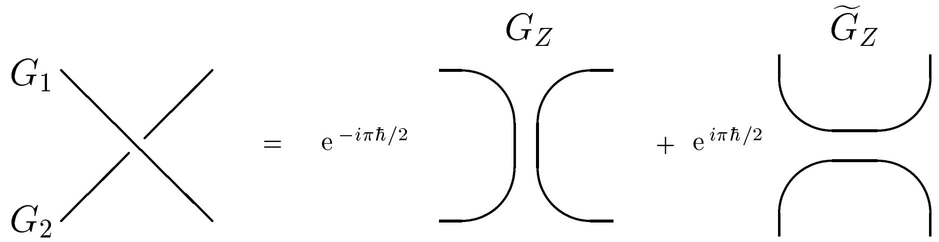
where

$$G_Z = \mathrm{tr}\,_1\,\mathrm{tr}\,_2\widehat{\xi}^1_{(ca)}\widehat{\xi}^2(bd)\big[e^1_{11}\otimes e^2_{22}+e^1_{22}\otimes e^2_{11}-e^1_{12}\otimes e^2_{21}-e^1_{21}\otimes e^2_{12}\big] \tag{3.26}$$

just like G_{I} in the classical case (2.17), whereas $\widetilde{G}_Z$ contains the quantum correction term,

$$
\begin{aligned}
\widetilde{G}_Z \;&=\; {}^{\times}_{\times}\,\mathrm{tr}\,_1\,\mathrm{tr}\,_2\widehat{\xi}^1_{(ca)}\widehat{\xi}^2(bd)(e^1_{ij}\otimes e^2_{ji})[X^1_Z\otimes X^2_Z]\,{}^{\times}_{\times} \\
&=\; :\mathrm{tr}\,_1\,\mathrm{tr}\,_2\widehat{\xi}^1_{(ca)}\widehat{\xi}^2(bd)(e^1_{ij}\otimes e^2_{ji})\big[X^1_Z\otimes X^2_Z+2(1-\cos\pi\hbar)e^1_{11}\otimes e^2_{22}\big]:.
\end{aligned}
\tag{3.27}
$$

Here $e^1_{ij}\otimes e^2_{ji}$ is the standard r-matrix that permutes the spaces "1" and "2," so, as a result, the "skein" relation of the form (2.17) appears. Locally, this relation has exactly the form proposed by Turaev [15], i.e., for intersecting geodesics, we have the defining relation



(3.28)

(The order of crossing G_1 and G_2 depends on which geodesic occupies the first place in the product.) So, we have ${}^{\times}_{\times} X^1_Z \otimes X^2_Z {}^{\times}_{\times} = X^1_Z \otimes X^2_Z + 2(1-\cos\pi\hbar)e^1_{11} \otimes e^2_{22}$, or, in 4×4-matrix form,

$$
{}^{\times}_{\times} X^1_Z \otimes X^2_Z {}^{\times}_{\times} = \left(\begin{array}{cccc} 0 & 0 & 0 & \mathrm{e}^{\,Z} \\ 0 & 2(1-\cos\pi\hbar) & -1 & 0 \\ 0 & -1 & 0 & 0 \\ \mathrm{e}^{\,-Z} & 0 & 0 & 0 \end{array} \right). \tag{3.29}
$$

Note, however, that assuming the geodesics G_1 and G_2 to be simple, we may turn the geodesics $\widetilde{G}_Z$ again into the simple geodesics $\widetilde{G}'_Z$ by performing the flip w.r.t. the edge Z as shown in Fig. 7.

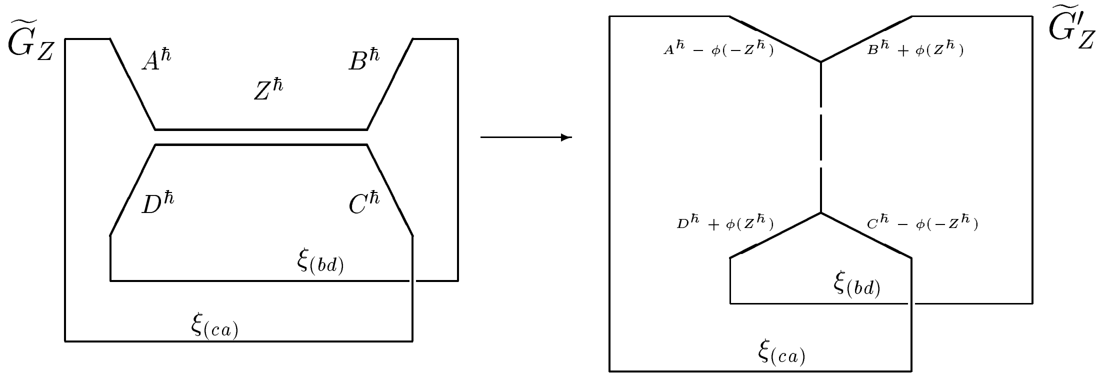


Fig. 7

The crucial point is to compare two inambiguously defined exrpessions: $\widetilde{G}'_Z$, which is to be Weyl ordered, and $\widetilde{G}_Z$ obtained from the geodesic algebra. Explicit calculations manifest that $\widetilde{G}_Z = \widetilde{G}'_Z$ and we formulate the following lemma.

Lemma 4 *The quantum geodesic ordering* ${}^{\times}_{\times}\ldots{}^{\times}_{\times}$ *generated by the geodesic algebra is consistent with the quantum modular transformations* (3.4), *i.e., the quantum geodesic algebra is modular invariant.*

4 Conclusion

In this paper, we briefly described how one may quantize the algebra of geodesics on moduli spaces of complex structures of Riemann surfaces. We see that one may construct all the variety of classical as well as quantum geodesics merely from a finite number of rather simple objects—the lengths of edges of the graph.

The next (and very intriguing) step is to construct representations of quantum geodesic algebras, which have to have nontrivial quantum-group-like structure for rational $\hbar = p/q$ and to compare the results with the ones obtained for quantum algebras of flat connections [16]. In our approach, when constructing such representations, we should take an advantage of our explicit realization of geodesic length operators, which are the proper quantum observables from the physical point of view.

We believe that all the construction above should find an application in physical problems, in particular, in the theory of effective action for $N=2$ SUSY Yang–Mills theory where a prepotential description has been elaborated recently. This deserves a detailed investigation.

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