Quantum chaos in optical models

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Abstract

Quasiclassical dynamics of a single atom obeying $(2j+1)-$ equidistant levels interacting with a quantized photon mode in an ideal cavity were considered. Corresponding equations of motion are received without the rotation-wave approximation. Chaotic behaviour is obtained in computer simulation for coupling constant values of the order of the atomic frequency.

1 Introduction

Recent technological achievements [1, 2, 3, 4] allows ones to investigate in practice the fundamental laws of quantum mechanics and quantum electrodynamics. Simplest models, which were still used in quantum optics only in thought experiments decade ago [5, 6, 7, 8], are realized in one-atom maser and optical microlaser devices. In these cases only two levels in the atom are taken in account. Indeed, near-resonant terms bring main contributions in transition probability between atomic levels. Moreover, standard perturbation theory cannot be applied if coupling constant reaches a value of the order of the atomic frequency. Research of the analogous many-atom system led to the chaotic behaviour in the semiclassical approximation [9, 10, 11, 12, 13].

Generally, there are some ways to define chaos in quantum systems. One of them consist in the analysis of quantum system parameters semiclassical dynamics. We will follow here this approach suggested in Refs. [14, 15].

The paper is organized as follows. In Sec. 2 model hamiltonians are presented. Sec. 3 contains a brief mathematical introduction of group theoretical coherent states (CS) and derivation of dynamic equations for their parameters. It is known, CS are quantum states the most close for classical ones. Because of this, CS basis is supposed to be quite relevant for quantum chaos researching. In Sec. 4 results of our computer simulations are considered and some generalizations of model hamiltonian and possibility of taking into account quantum corrections are discussed.

2 Model Hamiltonians

First of all we consider a single two - level atom, interacting with a single mode of quantized electromagnetic field near resonance. The Hamiltonian of this system can be represented as [5, 6, 7, 8]

$$\widehat{H} = \widehat{H}_F + \widehat{H}_A + (\,\widehat{H}_{int}^{(1)} + \widehat{H}_{int}^{(2)}\,)\,, \tag{1}$$

$$\widehat{H}_F = \omega\left(\widehat{b}^{+}\widehat{b} + \frac{1}{2}\right), \tag{2}$$

$$\widehat{H}_A = \omega_0\,\widehat{J}_0\,, \tag{3}$$

$$\widehat{H}_{int}^{(1)} = g\,\widehat{b}\widehat{J}_{+} + g\,\widehat{b}^{+}\widehat{J}_{+} + h.c.\,, \tag{4}$$

$$\widehat{H}_{int}^{(2)} = \kappa\,(\,\widehat{b}^{+} + \widehat{b}\,)^2\,, \tag{5}$$

here ω_0 is the atomic frequency (which equals difference between its energy levels, as we choose $\hbar = 1,$) $\omega\,(\approx \omega_0)$ is the frequency of the field, g and κ are coupling constants, $\widehat{b}^{+}$ and $\widehat{b}$ are the boson creation and annihilation operators, $\widehat{J}_0$ and $\widehat{J}_{\pm} = \widehat{J}_1 \pm \widehat{J}_2$ are one half of the corresponding Pauli matrices. Notice, then $\widehat{J}_{+}$ $(\widehat{J}_{-})$ increase (decrease) the energy of the atom by ω_0.

Standard Jaynes-Cummings model [5] is became by dipole and rotating-wave approximations. Firstly, $\widehat{H}_{int}^{(2)}$ is eliminated due to $\kappa \ll g$. Then, $g\,\widehat{b}^{+}\widehat{J}_{+}$ and hermitian conjugate term are treated as corresponding to high-frequency processes, which contribution discards for the macroscopic time. But it is not correct, when the coupling constant achieves values of the order of the atomic (or field) frequency. Here, the antiresonant terms' contribution is required to be taken into account. Recent experiments get hope to receive such (large) values of the coupling constant.

There is a number of the Jaynes - Cummings model (JCM) generalizations: one can consider the $(2j+1)$- level atom ("spin" $j = 1/2,\, 1,\, ...$), the case important in quantum electronics, and a case of m-quantum transitions between atomic energy levels when the direct transitions are forbidden. Here the Hamiltonian of interaction should be replaced by

$$\widehat{H}^{(1)}_{int} = g\,(\widehat{b})^m \widehat{J}_+ + g\,(\widehat{b}^+)^m \widehat{J}_+ + h.c.. \tag{6}$$

Operators J_0, $J_\pm$ in the Hamiltonian (6) and corresponding terms in (1) are treated as generators of $(2j+1)-$ dimensional representation of the group $SU(2)$.

3 Equations of motion

It has been noted in Introduction, the most natural way to study chaos in quantum systems consists in using the coherent states technique. If the Hamiltonian possesses dynamical symmetry group, this approach would be applied especial succesfully. A path-integral formulation of the problem leads to explicit motion equations for relevant group coherent states' parameters if the Hamiltonian is linear in generators of the dynamical group or approximate motion equations if it is not [16].

The direct product the Weyl group $W(1)$ and the $SU(2)$ group may be used as a dynamical group for our model Hamiltonian study. Let us introduce a direct product of corresponding coherent states as a coherent state of the system:

$$|CS> = |\alpha> \otimes |\zeta>, \tag{7}$$

here $|\alpha>$ is usual Glauber CS and $|\zeta>$ is "atomic" CS (ACS) [17, 18].

The set of CS possesses well-known properties (the resolution of unity is most important for our purpose).

A matrix element of the system evolution operator $\widehat{U}(t,t_0) = \exp[-i(t-t_0)\widehat{H}]$ between group - theoretical coherent states is represented as

$$< CS|exp[-i(t-t_0)\widehat{H}]|CS'> = \underbrace{\int \cdots \int}_{(N-1)} < CS_N|\exp[-i\,(t-t_0)\,\widehat{H}/N]|CS_{N-1}> d\mu(CS_{N-1}) \times$$

$$\times < CS_{N-1}|\exp[-i(t-t_0)\widehat{H}/N]|CS_{N-2}> \, d\mu(CS_{N-2}) \times \dots$$

$$\dots \times d\mu(CS_1) < CS_1|\exp[-i(t-t_0)\widehat{H}/N]|CS_0>, \qquad \Delta t = \frac{t-t_0}{N}, \quad N \gg 1, \tag{8}$$

$d\mu(CS_i)$ is the invariant measure in the resolution of unity. Then, we transform each factor in Eq.(8)

$$< CS_{j+1}\,|\exp[-i(t-t_0\widehat{H}/N]|CS_j> \simeq < CS_{j+1}\,|\hat{1} - i(t-t_0)\widehat{H}/N\,|CS_j> \simeq$$

$$\simeq < CS_{j+1}|CSj> \, \exp[-i(t-t_0)\,\mathcal{H}(CS_{j+1}|CS_j)],$$

$$\mathcal{H}(CS_{j+1}|CS_j) \equiv \frac{< CS_{j+1}|\widehat{H}|CS_j>}{< CS_{j+1}|CS_j>} \equiv \mathcal{H}(\alpha,\zeta|\alpha,\zeta)$$

is a covariant symbol of the Hamiltonian [19], and go over to limit by $N \to \infty$:

$$< \alpha,\zeta|\exp[-i(t-t_0)\widehat{H}]|\alpha\prime,\zeta\prime> = \int DM_\infty\, e^{i\mathcal{S}}\,.$$

Here $\mathcal{S}$ is treated as the action function.

Resulting evolution equations for the CS parameters takes the following form:

$$\begin{cases} \dot{\alpha} + i\,\nu\alpha + i\,2jgm(\alpha^*)^{m-1}\frac{\zeta}{1+\zeta\zeta^*} + i\,2j\lambda m(\alpha^*)^{m-1}\frac{\zeta^*}{1+\zeta\zeta^*} = 0 \\ \dot{\alpha^*} - i\,\nu\alpha^* - i\,2j\overline{g}m(\alpha)^{m-1}\frac{\zeta^*}{1+\zeta\zeta^*} - i\,2j\bar{\lambda}m(\alpha)^{m-1}\frac{\zeta}{1+\zeta\zeta^*} = 0 \\ \dot{\zeta} + i\,\omega\zeta + i\,\overline{g}(\alpha)^m + i\,\lambda(\alpha^*)^m - i\,g(\alpha^*)^m\zeta^{\,2} - i\,\overline{\lambda}(\alpha)^m\zeta^{\,2} = 0 \\ \dot{\zeta^*} - i\,\omega\zeta^* - i\,g(\alpha^*)^m - i\,\overline{\lambda}(\alpha)^m + i\,\overline{g}(\alpha)^m\zeta^{*\,2} + i\,\lambda(\alpha^*)^m\zeta^{*\,2} = 0. \end{cases} \tag{9}$$

Here the symbols * and $^-$ denotes the complex conjugation, the first one for the CS variables and the second one for the coupling costants.

Let us to introduce special notations for some particular cases of these equations: $g \neq 0\,, \lambda = 0$ (the standard JCM) or $g = \lambda$ (the "dressed" JCM) for $m = 1$ or $m = 2$.

Solutions of equations (9) must be found by reliable numerical method. The Dormand-Prince method of fifth order, which is concerned to Runge - Kutta Methods with automatic step control, was used in our computer program. The program permits us to get series of graphics: time evolution of real and imaginary parts of the CS parameters, their phase portraits and trajectories, complex - plane evolution of CS parameters both for field and atom, and transition probability time - dependence. For definiteness, the case to find atom in the "down" state is considered only:

$$P(t) = |<down|\zeta(t)>|^2. \tag{10}$$

Also, an evolution of the ACS was realized on the Bloch sphere: the complex parameter ζ is replaced by two angle parameters $\zeta = e^{i\phi}\cot(\frac{\theta}{2})$, that gives a stereographic map of a complex plane to the unit sphere [18]. The South Pole corresponds to the origin of the plane.

4 Results of computer simulations and discussion

Now, we discuss results of our computer calculations. First of all, the code was tested. So, standard JCM $(m = 1\,, \lambda = 0)$ has regular dynamics through all ranges of model parameters.

Account of the non-resonant terms leads to complication in the system behaviour. While at $g \leq 0.05 \div 0.1$ (a magnitude of the coupling constant depends on initial values of atomic and field coherent state parameters), the main contribution is comming from the resonant terms. Transition probability time dependence represents as a sine-like curve.

With growth g influence of anti-resonant terms increases and multifrequency regimes is observed. A number of frequencies concerns with average number of photons $< n >$. At $< n >\sim 1$, the regime is most close to the chaotic (for other equal conditions). The case $< n >\gg 1$ corresponds to classical field and the system dynamics is quite regular. However, might be found such value of g when a number of probability transition frequencies cuts down (to two or three, as we obtained). After this, multifrequency regime is returned and tends to chaos.

Beside, the case with two-quantum transitions was studied. Chaotic regimes arises at smaller values of the coupling constant as compared to the one-quantum transition. Notice, it would be expected that chaos has been demonstrated most clear at larger values g.

Pictures demonstrates some typical results.

In Fig.1, Fig.2 shown phase portraits of the real and imaginary parts of the ACS parameters at $m = 2$, $j = 2$ and $g = 0.2$. However, it is possible to generate the similar figures for any j and initial conditions and relevant g. The domains are fullfilled by points (of phase trajectory) and derivative increases very quickly: chaos is became. We get transition probability graphics and the Bloch sphere evolution (for the ACS), too.

Other two pictures in Fig.3 and Fig.4 represents an evolution of the ACS in the complex plane in case $j = 1$, and $j = 2$, respectively. Perhaps, they contains less information than phase portraits, but their whimsical forms deserve to show them.

Here we considered single atom dynamics in the semiclassical approximation only. As it was expected, the chaotic behaviour was best demonstrated by the system with neither very large quantum numbers nor very small. We suppose that quantum corrections to the dynamic equations (9) would lead to the same behaviour, but, at present, we consider these results as the first step of our investigation of chaos in quantum optical models.

The commonly used method of the Heisenberg operator averages [11, 12, 13] gives an opportunity to involve a wider class of problems because it does not requires an assumption about conservation subsystems' states as a coherent one during the evolution. Nevertheless, account of quantum corrections is believed to be more complicated technically in this approach. We compared the results in both approaches for the "dressed" Jaynes - Cummings model by calculating the inverted population difference. All of them are in a good agreement.

It is interesting to note, that the $SU(2)$ group generators may be replaced formally to the noncompact $SU(1,1)$ group generators. This substitution leads to well-known in nonlinear optics the three-wave parametric interaction dynamics [20]. Results of our computer simulations for quantum analogs of these models were published in [21].

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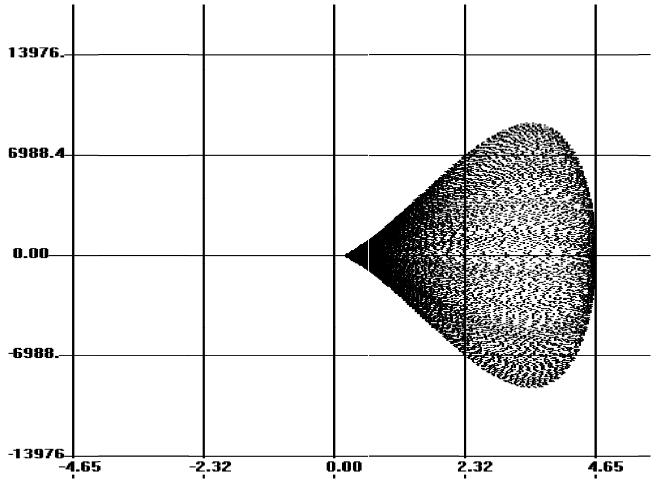

Figure 1: Phase portrait in the plane $(Re\,\zeta; Re\dot{\zeta})$; $(J = 2, m = 2, g = \lambda = 0.2, \quad 2\,\omega = \omega_0 = 1, Re\alpha(0) = 1, Im\alpha(0) = 0, Re\zeta(0) = 0.3, Im\zeta(0) = 0.)$

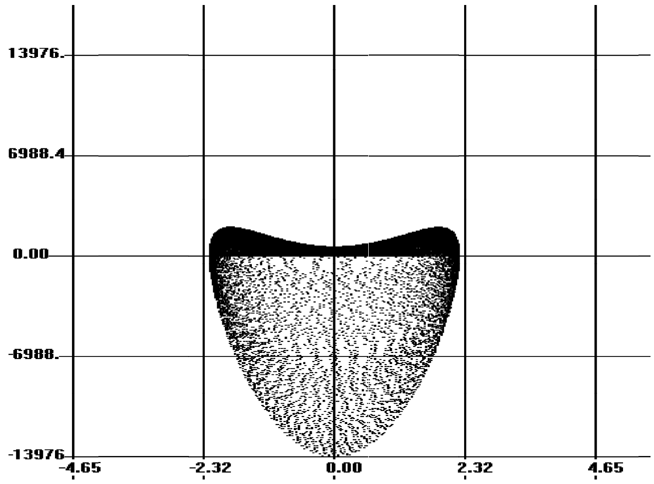

Figure 2: Phase portrait in the plane $(Im\,\zeta; Im\dot{\zeta})$; $(J = 2, m = 2, g = \lambda = 0.2, \quad 2\,\omega = \omega_0 = 1, Re\alpha(0) = 1, Im\alpha(0) = 0, Re\zeta(0) = 0.3, Im\zeta(0) = 0.)$

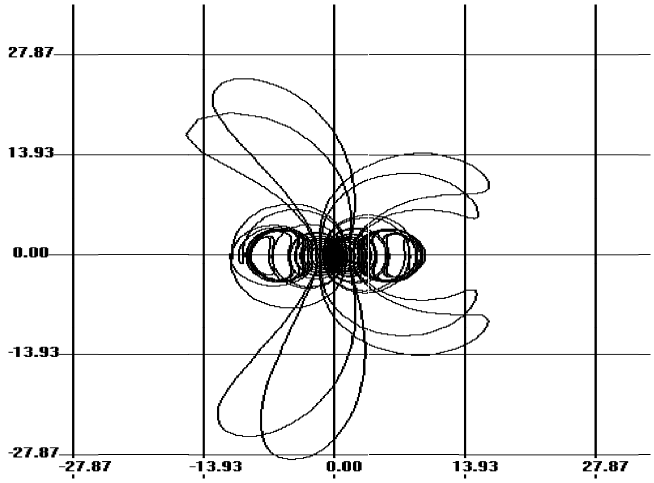

Figure 3: The atomic coherent state ("bee-like") trajectory in the complex plane $Re\zeta, Im\zeta$; $(J = 1,\, m = 1,\, g = \lambda = 0.2,\, \omega = \omega_0 = 1, Re\alpha(0) = Im\alpha(0) = 1,\ Re\zeta(0) = 0.5\ , Im\zeta(0) = 0)$.

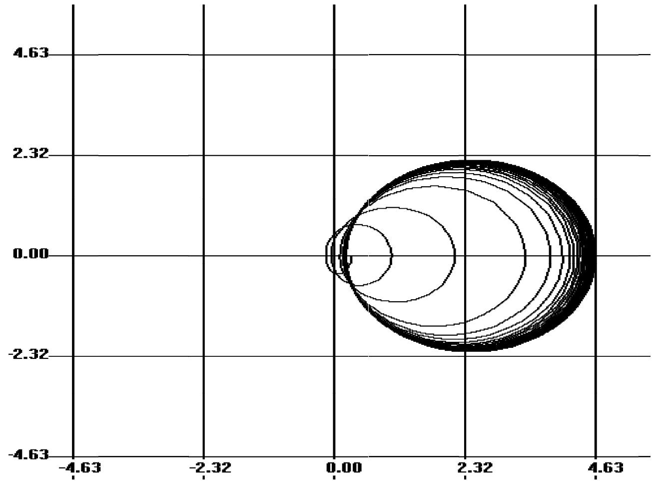

Figure 4: The atomic coherent state trajectory in the complex plane $Re\zeta, Im\zeta$; $(J = 2,\, m = 2,\, g = \lambda = 0.2,\, 2\,\omega = \omega_0 = 1, Re\alpha(0) = 1,\, Im\alpha(0) = 0,\ Re\zeta(0) = 0.3\ , Im\zeta(0) = 0)$.