Electric dipole in magnetic field: classical analogue of quantum BSEC and its quantization.

Nikita Sveshnikov* and Andrey Shirokov
Skobeltsyn Institute of Nuclear Physics, Moscow State University,
Moscow 119899, Russia

and
Derek Pursey
Department of Physics and Astronomy, Iowa State University,
Ames, IA 50011, U.S.A.

Abstract

We show that both rigid and non-rigid dipole can be trapped by an external uniform magnetic field in classical mechanics. The trapped states of the dipole present a non-trivial example of classical bound states embedded in continuum (BSEC) that can be treated as analogues of quantum BSECs. For example, the classical motion of the dipole is confined to a finite region in space though there are no classical turning points. We examine also the quantum motion of the dipole in magnetic field and show that for the most natural choices of the parameters (the rigid rotating dipole or the one bound by oscillator potential, uniform time-independent magnetic field, etc.) there are no quantum BSEC solutions. The possibilities of experimental investigations of BSECs are discussed.

1 Introduction

A quantum system can have bound states with normalizable wave functions and continuum spectrum (scattering) states with non-normalizable wave functions. It is usually supposed that the energies of all of the bound states $E_i < 0$ while the continuum spectrum states have the energies $E > 0$, i. e. the continuum spectrum states are well-separated in energy from the bound ones. However, as early as 1929 von Neumann and Wigner [1] have shown that a quantum system can have bound states embedded in continuum (BSEC), i. e. bound states with energies $E > 0$. Recently BSECs have been observed experimentally by F. Capasso et al [2]. The interest to BSECs was also stimulated by reported narrow e^+e^- coincidence peaks in heavy ion collisions at GSI, which were interpreted by Arbuzov et al [3] and independently by Vary and Spence [4] as due to bound states of the electron-positron system embedded in the continuum. So, BSECs have been studied in a number of recent papers [5]. Shirokov, Smirnov and Zaytsev [6] proposed also the so-called isolated states which are by definition the bound states that are not in correspondence with the S-matrix poles and which are a generalization of BSECs.

BSECs are conventionally regarded as exotic quantum states that do not have classical analogues. However, BSECs are very natural in classical mechanics. Really, if the potential energy of a classical system has two barriers than any state with positive energy trapped by the barriers is a BSEC. Nevertheless, the corresponding quantum system doesn't have a BSEC: a double-barrier potential generates resonance(s) instead of BSEC(s) in quantum mechanics.

To support a BSEC in quantum mechanics, a local potential should have an infinite number of oscillations with amplitudes decreasing at large distances [1, 7]. The wells of the long-range oscillating tail of the von Neumann-Wigner type potential are capable of trapping a classical particle with positive energy. Pursey and Weber [8] have analyzed this phenomenon in some detail. Note, however, that the quantum BSEC wave function in this case decreases very slowly as the distance r tends to infinity. As

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a result, the probability to detect the particle outside the potential well that can trap it, is much larger than the probability to detect it within the well. More, the quantum BSEC in the von Neumann-Wigner type potential can have an energy larger than the maximum of the potential energy, so, the corresponding classical motion doesn't have classical turning points. Therefore, the von Neumann-Wigner type potential doesn't provide an example of the classical analogue of the quantum BSEC.

BSECs are natural in quantum mechanics in the case when the interaction is non-local [6, 9] or in the case of multichannel scattering (see, e. g., [10, 11]). However, such quantum systems do not have explicit classical analogues.

So, when a quantum system has a BSEC, the corresponding classical system doesn't and vice versa. The reason for this is that the mechanisms of the formation of BSEC in quantum and classical mechanics are very different: in the classical case the motion of a particle with a positive energy is confined to a finite region of space due to potential barriers that give rise to the classical turning points while in the quantum case the BSEC wave function decreases at large distances due to interference of the waves reflected by an infinite number of potential barriers in the case of the von Neumann–Wigner type potential or due to the features peculiar to the quantum non-local interaction or multichannel scattering. Thus, it would be interesting to find a classical system that supports BSEC(s) with no classical turning points where the trapping of the particle is not associated with potential barriers. Such classical BSECs can serve as analogues of quantum BSECs and probably visualize the formation of quantum BSECs.

Classical BSECs have been discussed by Pollak and Pechukas [12] in the context of reactive scattering theory in physical chemistry. Pollak and Pechukas have shown that a classical system can have a BSEC solution in the case when the potential surface has a saddle point. There is a close analogy between the Pollak and Pechukas BSEC solutions and the position of unstable equilibrium of a physical pendulum. The unstable equilibrium of a classical system when the particle is stopped for an infinite time just at the maximum point of the potential barrier presents some kind of a classical BSEC. However, such BSEC is uninteresting because (i) it is unstable and any small perturbation destroys it and (ii) the corresponding quantum system doesn't have a BSEC solution at the same or close energy.

Recently we suggested [13] another example of a classical BSEC that seems to have some features of quantum BSECs. In particular, we demonstrated [13] that a classical rigid electric dipole can be trapped by an external magnetic field. The trapping of the dipole is not associated with any potential, thus this system presents a non-trivial example of a classical BSEC without classical turning points. However, we did not examined the quantum solutions for the dipole in the magnetic field. Thus, it is not clear whether the electric dipole in magnetic field presents the first example of a system that has BSEC solutions both in quantum and classical cases. We discuss this problem in this paper. First, we present the classical equations for the dipole in magnetic field in the form that can be easily generalized on the quantum case, and prove the existence of the classical BSEC solutions. Contrary to the Ref. [13], we do not discuss the general case of the classical motion of the rigid dipole in magnetic field but restrict the discussion to a particular case that can be solved analytically. In addition we present the equations of classical motion and demonstrate the existence of classical BSEC solutions for a particular solvable case of a non-rigid dipole in magnetic field. Next, we turn to the quantum consideration of the above examples.

2 Classical Trapping of Electric Dipole by Magnetic Field

2.1 Rigid rotating dipole

Our model of the rigid dipole is a massless rod fixing the distance $2a$ between two particles of the mass m each. One of the particles carries the charge $+e$ while the charge of the other is $-e$. The uniform magnetic field $\boldsymbol{B}$ is supposed to be parallel to the Y axis. We restrict the center of mass motion of the dipole to the Z direction and the rotational degrees of freedom of the dipole to the XY plane. The general case of the classical motion of the rigid dipole in magnetic field is discussed in Ref. [13].

The Hamiltonian in our case is the following:

$$H = \frac{1}{4m}(P + \beta a \cos\varphi)^2 + \frac{1}{4ma^2}(\pi + \beta R a \sin\varphi)^2, \tag{1}$$

where R is the center of mass coordinate, P is the canonically conjugate momentum, φ is an angle between the X axis and $\boldsymbol{r}$ with $\boldsymbol{r}$ being the relative coordinate, $\boldsymbol{r} = \boldsymbol{r_1} - \boldsymbol{r_2}$, where $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ are the coordinates of the particles, π is the momentum canonically conjugated to φ, and $\beta = eB/c$.

It is easy to check that there is an integral of motion,

$$J = P - \beta a \cos\varphi. \tag{2}$$

It is important to note that J is the integral of motion in the quantum case, because, as it is seen from Eqs. (1) and (2), the commutator $[\hat{H}, \hat{J}] = 0$ (hereafter we use hat to denote operators). Of course, it is also true in the classical case, and the corresponding Poisson bracket $\{H, J\} = 0$.

The canonical classical equations of motion are

$$\dot{R} = \frac{1}{2m}(P + \beta a \cos\varphi)\,, \tag{3}$$

$$\dot{\varphi} = \frac{1}{2ma^2}(\pi + \beta R a \sin\varphi)\,, \tag{4}$$

$$\dot{P} = -\frac{\beta}{2ma}(\pi + \beta R a \sin\varphi)\sin\varphi\,, \tag{5}$$

$$\dot{\pi} = \frac{\beta}{2m}\left[aP\sin\varphi - \frac{R}{a}\,\pi\cos\varphi - \beta(R^2 - a^2)\sin\varphi\cos\varphi\right]. \tag{6}$$

From Eqs. (3)—(6) it is easy to derive the equations of motion in the form

$$2m\ddot{R} = -2\beta a\dot{\varphi}\sin\varphi\,, \tag{7}$$

$$2ma^2\ddot{\varphi} = \beta a\dot{R}\sin\varphi\,. \tag{8}$$

The right-hand sides of Eqs. (7) and (8) are obviously the net force and the net torque, respectively, applied to the dipole.

Using (2), equation (3) can be rewritten as

$$\dot{R} = \frac{1}{2m}(J + 2\beta a\cos\varphi)\,. \tag{9}$$

The total energy of the system E is the kinetic energy. Thus, using Eq. (9) the total energy can be expressed as

$$ma^2\dot{\varphi}^2 + \frac{1}{4m}(J + 2\beta a\cos\varphi)^2 = E. \tag{10}$$

The classical bound (or trapped) state corresponds to the case when the mean value

$$\langle\dot{R}\rangle_t = 0. \tag{11}$$

So, classical BSECs are the solutions of Eq. (10) that fit the following equation:

$$\langle\dot{R}\rangle_t = \frac{1}{2m}\langle J + 2\beta a\cos\varphi\rangle_t = \frac{1}{2mT}\int_0^T (J + 2\beta a\cos\varphi)\,dt = 0\ , \tag{12}$$

where T is a period of motion.

If $J < -2\beta a$ or $J > 2\beta a$, then it is seen from (12) that $\langle\dot{R}\rangle_t \neq 0$, and the trapping is impossible. So, let us examine the case $|J| \leq 2\beta a$.

Suppose $0 < J \leq 2\beta a$. From Eq. (10) we have:

$$\frac{d\varphi}{dt} = \sqrt{\frac{E}{ma^2} - \frac{1}{4(ma)^2}(J + 2\beta a\cos\varphi)^2}\ , \tag{13}$$

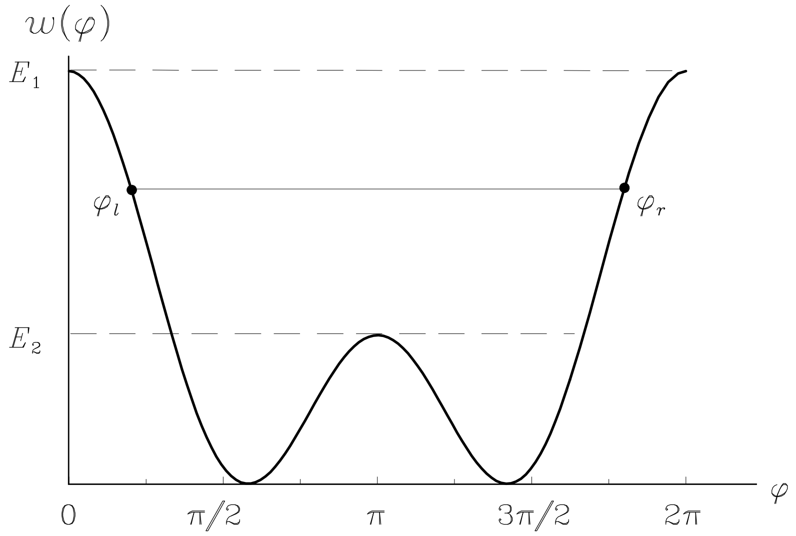

Figure 1: Effective potential $w(\varphi)$ vs φ for $0 < J < 2\beta a$.

or

$$dt = \frac{\sqrt{m}\, a\, d\varphi}{\sqrt{E - \frac{1}{4m}(J + 2\beta a \cos\varphi)^2}} \,. \tag{14}$$

Equation (10) describes the motion of a particle in the effective potential $w(\varphi) = \frac{1}{4m}(J + 2\beta a \cos\varphi)^2$ (see Fig. 1). The classical motion is confined between the left, φ_l, and the right, φ_r, classical turning points. Thus, Eq. (12) with the help of (14) is reduced to

$$\langle \dot{R} \rangle_t = \frac{a}{\sqrt{m}\, T} \int\limits_{\varphi_l}^{\varphi_r} \frac{(J + 2\beta a \cos\varphi)\, d\varphi}{\sqrt{E - \frac{1}{4m}(J + 2\beta a \cos\varphi)^2}} \,. \tag{15}$$

The effective potential $w(\varphi)$ has two maxima, $E_1 = \frac{1}{4m}(J + 2\beta a)^2$ at $\varphi = 0$ and $E_2 = \frac{1}{4m}(J - 2\beta a)^2$ at $\varphi = \pi$. When $E = E_1$, the motion is aperiodic, and $\langle \dot{R} \rangle_t \to \infty$ as $E \to E_1$. The motion is also aperiodic if $E = E_2$. However, in this case $\langle \dot{R} \rangle_t \to -\infty$ as $E \to E_2$, because $J + 2\beta a \cos\varphi < 0$ in the vicinity of $\varphi = \pi$. $\langle \dot{R} \rangle_t$ is a continuous function of E that is negative for $E = E_2 + \varepsilon$ and positive for $E = E_1 - \varepsilon$. Thus, $\langle \dot{R} \rangle_t = 0$ for some $E \in (E_1, E_2)$, and the motion is trapped. Omitting the details, we just note, that the analysis of (15) shows, that the motion is untrapped for any $E > E_1$ and for any $E < E_2$ with the only trivial exception $E = 0$.

The case of $-2\beta a \leq J < 0$ can be analyzed in the same manner. The case $J = 0$ is trivial, surely, $\langle \cos\varphi \rangle_t = 0$ in this case, and the motion is trapped for any E with the only exception $E = E_1$ when it is aperiodic.

Thus, our conclusions are the following:

(i) If $J = 0$, then the classical motion is trapped for any energy $E > 0$ with the only exception of $E = \beta^2 a^2/m$ when the motion is aperiodic;

(ii) For $E > \beta^2 a^2/m$ trapped state does not exist for any $J \neq 0$.

(iii) For any E from the interval $0 < E < \beta^2 a^2/m$ there exists $J(E) \in (0, 2\beta a)$ such that for $J = \pm J(E)$ the classical motion is trapped. $J(E) \to 0$ when $E \to \beta^2 a^2/m$, and $J(E) \to 2\beta a$ when $E \to 0$.

A qualitative plot of the curve $J(E)$ showing the combinations of values of the energy E and of the integral of motion J corresponding to classical BSECs, is presented on Fig. 2.

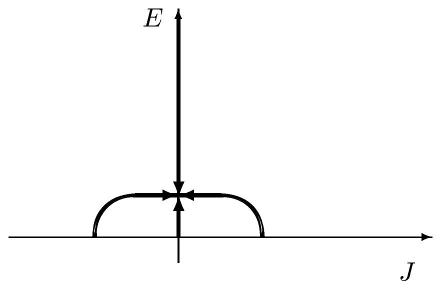

Figure 2: A qualitative plot of $J(E)$ on the plane E vs J for the rigid dipole. Thick lines show the values of J that for a given E give rise to the trapped motion.

2.2 Non-rigid dipole

Our model of the non-rigid dipole is two particles of the mass m each bounded by the potential $W(\boldsymbol{r})$ where $\boldsymbol{r}$ is the relative coordinate, $\boldsymbol{r} = \boldsymbol{r_1} - \boldsymbol{r_2}$. One of the particles carries the charge $+e$ while the charge of the other is $-e$. The uniform magnetic field $\boldsymbol{B}$ is supposed to be parallel to the Y axis. The Y projection of the net force experienced by the dipole is zero, therefore, the motion of the center of mass of the dipole in Y direction is of no interest and can be treated independently. So, we can chose Z axis in the direction of the center of mass motion. We restrict our discussion to the case when the relative coordinate $\boldsymbol{r}$ is always perpendicular to the center of mass velocity $\dot{\boldsymbol{R}}$, i. e. to the Z axis.

The Hamiltonian is

$$H = \frac{1}{4m}\left(P + \frac{1}{2}\beta x\right)^2 + \frac{1}{m}\left(p_x - \frac{1}{2}\beta R\right)^2 + \frac{1}{m}p_y^2 + W(x,y), \tag{16}$$

where x and y are the projections of the relative coordinate $\boldsymbol{r}$ on X and Y axes, respectively; p_x and p_y are the momenta canonically conjugated to x and y, respectively, and $W(x,y)$ is the potential energy of the relative motion of the particles.

There is an integral of motion,

$$J = P - \frac{1}{2}\beta x. \tag{17}$$

As in the case of the rigid dipole, J is an integral of motion both in the classic and quantum cases.

In order to simplify the problem, we make some assumptions concerning the potential $W(x,y)$ binding the particles. In particular, we suppose that the x and y dependences of $W(x,y)$ can be separated, i. e. $W(x,y)$ can be expressed as $W(x,y) = V(x) + \tilde{V}(y)$. This is the case, for example, of the the harmonic oscillator potential $W(x,y)$. Due to this assumption the motion in the y direction can be treated independently. We also can get rid from the motion in y direction supposing that the dipole is constrained to the XZ plane. So, in what follows we shall discuss the simplified Hamiltonian

$$H = \frac{1}{4m}\left(P + \frac{1}{2}\beta x\right)^2 + \frac{1}{m}\left(p_x - \frac{1}{2}\beta R\right)^2 + V(x). \tag{18}$$

The canonical equations of motion associated with the Hamiltonian (18) are

$$\dot{R} = \frac{1}{2m}\left(P + \frac{1}{2}\beta x\right), \tag{19}$$

$$\dot{x} = \frac{2}{m}\left(p_x - \frac{1}{2}\beta R\right), \tag{20}$$

$$\dot{P} = \frac{\beta}{m}\left(p_x - \frac{1}{2}\beta R\right), \tag{21}$$

$$\dot{p}_x = -\frac{\beta}{4m}\left(P + \frac{1}{2}\beta x\right) - \frac{dV(x)}{dx}\,. \tag{22}$$

From Eqs. (19)—(22) it is easy to derive the equations of motion in the form

$$2m\ddot{R} = \beta\dot{x}\,, \tag{23}$$

$$\frac{m}{2}\,\ddot{x} = -\beta\dot{R} - \frac{dV(x)}{dx}\,. \tag{24}$$

Eqs. (23) and (24) are, obviously, Newton equations for the motion of the center of mass and for the relative motion of the particles.

Using (17), equation (19) can be rewritten as

$$\dot{R} = \frac{1}{2m}(J + \beta x)\,. \tag{25}$$

The total energy of the system E is the kinetic energy. Thus, using Eq. (25) the total energy can be expressed as

$$\frac{m}{4}\,\dot{x}^2 + \frac{1}{4m}(J + \beta x)^2 + V(x) = E\,. \tag{26}$$

For the trapped motion, $\langle\dot{R}\rangle_t = 0$, and using (25) we get

$$J + \beta\langle x\rangle_t = 0\,. \tag{27}$$

$\langle x\rangle_t$ that should be derived from the Eq. (26), is a function of E and J. For an arbitrary potential $V(x)$, Eq. (27) may have or have not solutions. Let as discuss few particular choices of $V(x)$.

First, let us note that if $J = 0$ then for any symmetric potential[1] $V(x) = V(-x)$ such that the effective potential $w(x) = V(x) + \dfrac{\beta^2 x^2}{4m}$ is a monotonic increasing function for $x > 0$, we have the trapped state at any energy E.

Next, let us discuss the case of the oscillator potential, i. e., $V(x) = \dfrac{m\omega^2}{4}(x - x_0)^2$. It is easy to derive from (26) that in this case,

$$\langle x\rangle_t = \frac{m^2\omega^2 x_0 - \beta J}{\beta^2 + m^2\omega^2}\,. \tag{28}$$

From Eqs. (27) and (28) we obtain, that the motion for any energy E is trapped if and only if

$$J = -\beta x_0\,. \tag{29}$$

Let us also discuss the case of the potential that has a plateau at some interval of x, i. e., we suppose that $V(x) = V_0 = \text{const}$ for $x \in [-L + x_0,\, L + x_0]$. If $J \in (-\beta x_0 - \beta L,\, -\beta x_0 + \beta L)$, then the effective potential $w(x) = V(x) + \dfrac{(J + \beta x)^2}{4m}$ is a parabola which has a minimum at $x = -J/\beta$. If the energy E lies in the vicinity of the minimum, i. e. both classical turning points belong to the parabola, $\langle x\rangle_t = -J/\beta$ and $\langle\dot{R}\rangle_t = 0$. Thus, we conclude that the classical motion is trapped for any energy $E \in \left[V_0,\, V_0 + \dfrac{\beta^2 L^2}{4m}\right]$ provided that J takes any value from the interval
$J \in \left[-\beta(L + x_0) + \sqrt{4m(E - V_0)},\ \beta(L - x_0) - \sqrt{4m(E - V_0)}\,\right]$ (see Fig. 3).

This example is very interesting, because it shows that the dipole can be trapped in a wide and continuous range of the values of the energy E and of the integral of motion J, or, in other words, in a wide and continuous range of initial conditions.

[1] Of course, the potential binding the particles should be symmetric. However, the case of the asymmetric potential may be also of interest, because we can subject the dipole to the influence of the external electric field in the X direction. In this case, we can use the Hamiltonian (18) with $V(x) = V_0(x) + \Phi(x)$, where $\Phi(x)$ is the potential of the external electric field.

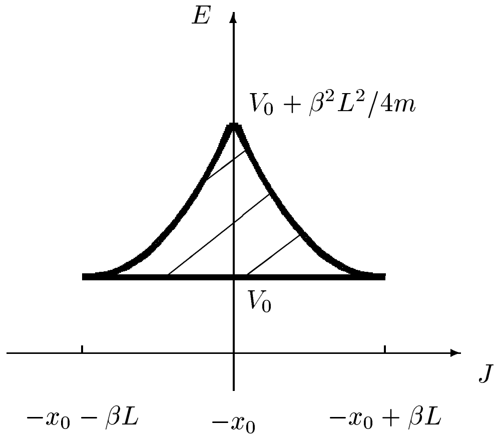

Figure 3: The shaded area shows qualitatively the region of E and J that gives rise to the trapped motion in the case of the potential with the plateau.

3 Quantization

For both problems we have a Hamiltonian and an additional integral of motion commuting with the Hamiltonian. So, in the quantum case, we should search for a common solution of the Schrödinger equation

$$\hat{H}\Psi = E\Psi, \tag{30}$$

where $\hat{H}$ is given by (1) or (18), and equation for the integral J,

$$\hat{J}\Psi = J\Psi, \tag{31}$$

where the operator $\hat{J}$ is given by (2) or (17).

3.1 Rigid dipole

The following trick can be used to simplify the Hamiltonian (1). Let us introduce the wave function ψ that is connected with Ψ according to

$$\Psi = e^{i\beta Ra\cos\varphi}\psi \tag{32}$$

(hereafter we are using the system $\hbar = 1$). Using Eqs. (1) and (2), we reduce (30) and (31) with the help of (32) to

$$\left[\frac{1}{4m}(\hat{P} + 2\beta a\cos\varphi)^2 + \frac{1}{4ma^2}\,\hat{\pi}^2\right]\psi = E\psi \tag{33}$$

and

$$\hat{P}\psi = J\psi. \tag{34}$$

Eq. (34) is seen to have a continuum spectrum with plane waves $\exp(iJR)$ been the solutions. $\hat{P}$ entering (33) can be substituted by its eigen value J, and this equation can be solved with respect to the single variable φ. Eq. (33) is subjected to the periodic boundary conditions, $\psi(\varphi) = \psi(\varphi + 2\pi)$, thus, the spectrum of E for any given J is discrete. An infinite number of discrete energy eigen values $E_n(J)$ corresponds to any particular value of J. However, the lowest eigen energy $E_0(J) \sim J^2$ for large J. Thus, any finite energy E can be realized only by a finite number of the plane waves with a given momentum J. It is impossible to construct a normalizable wave function as a superposition of a finite number of plane waves. Therefore, the rigid dipole in a uniform magnetic field does not have quantum BSEC solutions.

3.2 Non-rigid dipole

Applying to Eqs. (30) and (31) with $\hat{H}$ given by (18) and $\hat{J}$ given by (17) the same trick as in the case of the rigid dipole, we obtain equations

$$\left[\frac{1}{4m}(\hat{P}+\beta x)^2+\frac{1}{m}\hat{p}_x^2+V(x)\right]\psi=E\psi \tag{35}$$

and

$$\hat{P}\psi=J\psi \tag{36}$$

for the function

$$\psi=e^{-\frac{i\beta xR}{2}}\Psi. \tag{37}$$

Due to (36), operator $\hat{P}$ in (35) can be replaced by its eigen value J.

The most natural choice for the potential $V(x)$ is the oscillator potential, $V(x)=\frac{1}{4}m\omega^2(x-x_0)^2$. The Schrödinger equation (35) in this case can be easily solved yielding for any value of the integral J an infinite number of discrete eigen values

$$E_n=\sqrt{\omega^2+\frac{\beta^2}{m^2}}\left(n+\frac{1}{2}\right)+\frac{1}{4}m\omega^2x_0^2+\frac{1}{4m}J^2-\frac{(\beta J-m^2\omega^2x_0)^2}{4m(m^2\omega^2+\beta^2)}\,. \tag{38}$$

However, for any particular value of E we have only a finite number of the plane waves with a given momentum J. Thus, there are no quantum BSEC solutions for the dipole bound by the oscillator potential, too.

4 Discussion.

We have shown that a classical dipole can be trapped by magnetic field. The trapping is possible in all cases considered provided that the integrals of motion E and J take some particular values. The trapping of the dipole presents an interesting and non-trivial example of a classical BSEC. This classical BSEC has some features inherent to quantum BSECs, in particular, the trapping of the dipole is not associated with potential barriers and classical turning points.

We have also investigated the corresponding quantum problem. We have found out that there are no quantum BSECs for the most natural choices of the parameters of the system (the rigid rotating dipole or the one bound by oscillator potential, uniform time-independent magnetic field, etc.). This conclusion looks strange and intriguing. Really, as we have already mentioned in the Introduction, all known quantum systems supporting BSEC(s) either do not have classical analogue (e. g., systems with non-local interaction) or the corresponding classical system doesn't support BSEC. On the other hand, from all known examples of classical systems supporting BSECs we can conclude that the corresponding quantum system doesn't have BSEC solutions, and our study seems to confirm this conclusion even for classical systems that support BSECs without classical turning points.

One can suppose that in the quantum case the dipole have no BSEC solutions due to the uncertainty principle. Really, the classical BSEC solutions for the dipole are possible only for certain values of the integrals of motion E and J, i. e. the conditions for the classical BSEC involve a very special relation between the initial values of the variables and their time derivatives. So, one can suppose that when translated into variables and their canonical momenta, the condition for a classical BSEC will require a constraint relation on the initial values of both variables and their momenta. It is not at all clear that a constraint of this type could be imposed in the quantum theory without violating the uncertainty principle.

Note, however, the example of the classical potential with plateau discussed above (see Fig. 3). As we have shown, in this case the dipole is trapped in a wide and continuous range of E and J, so, the initial conditions should not violate the uncertainty principle.

So, quantum BSEC solutions may be possible if the dipole is bound by some potential other than the harmonic oscillator potential. The experience we get pushes us to search for a potential $V(x)$ that will allow for an infinite number of continuous J eigen values fitting the same energy.

Two limiting cases for such a potential we derive directly from (38). The first possibility is the limit $m \to \infty$. This limit is equivalent to the limit $\omega \to \infty$, or, more accurately, $\omega \gg \beta/m$, thus, this is the limit of a weak magnetic field. Generally, this limit seems to be just the classical limit, and one can expect quantum BSECs in this limit equivalent to the classical ones discussed above. However, the situation is not so simple. Even in the $m \to \infty$ limit the motion in the lowest oscillator states remains essentially quantum. Thus, the quantum BSECs are expected in the limit $m \to \infty$ and $n \to \infty$. In the limit $m \to \infty$ and $n \sim 1$ only the center of mass motion is classical while the internal oscillations are quantum. Probably, BSEC solutions can be found in the R-classical+x-quantum limit, but we have not examined this limit thoroughly yet.

Another possibility is the opposite limiting case of $\omega \to 0$ that brings us to the trivial situation with $V(x) \equiv 0$. In this case, we should obtain BSECs corresponding to the rotation of two non-interacting charged particles in the magnetic field. The solutions of Eq. (35) for $V(x) \equiv 0$ are seen to be the harmonic oscillator wave functions $\varphi_n(x + J/\beta)$ of the frequency $\omega_0 = \beta/m$, and the total wave functions

$$\Psi = e^{\frac{i\beta xR}{2}} \psi = e^{\frac{i\beta xR}{2}} e^{iJR} \varphi_n(x + J/\beta). \tag{39}$$

These functions are seen to have a non-zero flux of the center of mass motion. Thus, J is a bad quantum number for BSECs. To obtain a BSEC, we should integrate the wave function (39) over J:

$$\Psi_{BSEC} = \int e^{\frac{i\beta xR}{2}} e^{iJR} \varphi_n(x + J/\beta)\, dJ = \text{const} \cdot e^{-\frac{i\beta xR}{2}} \varphi_n(R), \tag{40}$$

where we make use of the well-known fact (see, e. g., [14]) that the Fourier transformation of the oscillator function $\varphi_n(R)$, i. e., the oscillator wave function in momentum space, is up to a constant the same oscillator function $\varphi_n(q)$ of the new variable q.

Thus, for the simple and non-interesting case we managed to derive a BSEC solution (40). It would be interesting to find a non-trivial and physically reasonable potential $V(x)$ binding the particles that will provide BSEC solutions for the dipole in magnetic field.

Note, that it is possible to search for such potential $V(x)$ using the same approach that we have utilized in the classical case, i. e., solving Eqs. (11) and (27). In the quantum case, $\dot{R} = i[\hat{H}, R]$. If Ψ is a normalizable BSEC wave function, then

$$\langle \Psi \mid \dot{R} \mid \Psi \rangle = i \langle \Psi \mid [\hat{H}, R] \mid \Psi \rangle = i\left(E \langle \Psi \mid R \mid \Psi \rangle - \langle \Psi \mid R \mid \Psi \rangle E\right) = 0. \tag{41}$$

If Ψ is a non-normalizable continuum spectrum wave function, then $\langle \Psi \mid R \mid \Psi \rangle$ diverges and Eq. (41) is not valid. Calculating the commutator $[\hat{H}, R]$, we immediately obtain Eq. (27) from (41). Thus, we can search for the potential $V(x)$, or, better to say, for the effective potential $w(x) = V(x) + \dfrac{1}{4m}(J + \beta x)$ that gives rise to wave functions fitting Eq. (27). The above treatment shows that we should search for the potential supporting solutions fitting Eq. (27) at least on a finite interval of the continuous variable J.

We believe it will be possible to find such physically reasonable potential. Note, that potential $V(x)$ binding, say, a dipole molecule can be modified to some extend by an external electric field in X direction. We suppose that to find the potential $V(x)$ giving rise to quantum BSEC solutions for the dipole, will be valuable because (i) it will demonstrate that BSECs can be present in a system that is very simple and involves only well-known local interactions; (ii) it will present an example of a system that has both classical and quantum BSECs; (iii) we hope it will be possible to find a real system supporting BSEC(s) that can be studied experimentally.

However, the rigid and non-rigid dipoles that we have examined in this paper seem also to be interesting for experimental investigation. Really, a dipole molecule should behave classically in a weak magnetic field, but in a stronger field quantum effects should become important. So, in a gas of dipole molecules some of them should be trapped by the weak magnetic field. When the field is increased, the trapped state should be destroyed due to quantum effects. We suppose that this effect never discussed before in literature can be investigated experimentally.

Classical BSEC destroyed by the magnetic field strong enough to give rise to quantum effects should turn into a very narrow long-living resonance of a very specific type. Such resonances will be also interesting to study experimentally. These resonances should effect strongly the density of states in the system and give rise to some peculiarities in dispersion of some excitations.

BSECs discussed in this paper may be also interesting for the studies of chaotic motion. The general classical Hamiltonian of the rigid dipole is non-integrable [13]. The same is true for the non-rigid dipole. Therefore, the motion of the dipole can be chaotic. The classical BSEC solutions correspond to the isles of integrability in the parameter space of the system. It would be interesting to study the influence of such isles of integrability on the chaotic motion in the system. Of course, it would be even more interesting to extend the chaotic studies on the corresponding quantum case.

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