Boundary Effects and Confinement in the Gauge Field Theory

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Abstract

The confinement criterion based on the dependence of the partition function on the boundary condition for the electric field strength is presented for the non–abelian gauge theory. It is supported by an approximate calculation of the partition function and the Wilson loop for SU(N) gluodynamics in the 3-d Fock–Schwinger gauge. Possible directions of further studies of confinement are outlined.

1 Introduction

It was a great privelege for me to be one of Nikita Alexeevich's students. Our acquaintance began in 1989 and in 1995 I have defended my PhD thesis under his supervision on the subject of this paper. His bright personality and elegant scientific style had a major impact on formation of my own approach to theoretical physics. It is with deep sadness and a feeling of irrecovable loss that I am writing this brief overview of our unfinished work.

The problem of confinement in the non–abelian gauge theory — one of the great challenges of the modern particles physics — particularly facinated Nikita Alexeevich over many years. He continued working on it in the last few years when my research area has changed towards the non–equilibrium statistical mechanics.

N.A. Sveshnikov was a big experts in the infrared divergencies in the gauge theories. He had a strong belief that the confinement phenomenon is closely connected to IR properties of the Yang–Mills theory. He also felt that somehow the surface terms play a crucial role in it and that the problem may be rather obscured by ambigous gauge fixing. Having got these insights into the underlying physics from his previous experience in the asymptotic dynamics, Nikita Alexeevich's approach was direct and aimed at the heart of the problem.

First of all, one has to regularise IR divergences by enclosing the system into a compact space–time domain. This practically means taking a finite space volume V and limiting the Eucledean time in the path integral to a finite inverse temperature β range. The latter has an added value of permitting study of the thermodynamics of the confinement–deconfinement transition. Next, one has to be careful with the choice of the gauge condition. Covariant gauges have a danger of time–dependent residual transformations and they are unwieldly at a finite temperature anyway. The gauge which I was suggested by N.A. Sveshnikov and Edward Boos to consider in 1989, namely the so–called 3-d Fock–Schwinger gauge [1], is remarkable in that it permits explicit solution of the Gauss law constraint. This produces a rather simple polynomial, though a non–local, Hamiltonian in terms of the dynamical variables. Most importantly, due to the spherical symmetry it is straightforward to impose boundary conditions on these variables. The dynamics of volume variables appears quite elegant, although it is intermingled with the dynamics of surface variables (aka variables at infinity [2] in the limit of infinite volume).

Naturally, there are some conserved variables (integrals of motion) which can be constructed out of the surface variables. These normally are some kinds of order parameters in statistical mechanics. The important question is then how the partition function of the system depends on these conserved parameters in the thermodynamic limit. Such is the route a logical mind would try to follow in an attempt to understand confinement from the first principles of the fundamental Yang–Mills theory, rather than looking for phenomenological ways of modelling it.

Quite often in statistical mechanics finding the proper order parameter controlling a particular phase transition means winning a half of the battle. With a bit of simple algebra we can demostrate that the integrals of motion we are talking about here are in fact the fluxes of the colour in all possible spatial directions at infinity. Now the intuition pays off — in the confinement phase all such fluxes should vanish to prohibit any colour to escape from a localised interaction region. Therefore, confinement means that colour states could not exist as asymptotic states. On the contrary, in the deconfinement phase no such

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strong restriction is present and colour states are experimentally observable. Thus, to prove confinement one merely has to show that the statistical mean values of the order parameters exhibit precisely this behaviour with changing temperature.

Bearing this in mind there is still a long way off to calculate the partition function and the mean values in such a non–linear theory as the Yang–Mills theory in any sufficiently accurate approximation. The perturbative treatment is apparently of no use as we believe the phenomena to be essentially non–perturbative.

There are two standard approximation which are universally adopted in statistical mechanics in difficult situations like this. One is the Bogoliubov variational treatment. Another, a less ambitious, but much simpler approach is widely known as the mean–field approximation. Being clever and applying the mean–field method in terms of the right collective variables one can describe rather complex phenomena and extract many new predictions. In any case, we believe that we have succeeded in finding the "good" mean–field expression for the partition function here by exactly transforming the path integral to the gauge strengths variables and applying the stationary phase approximation. What we have found is, first of all, rather non–perturbative, what is manifested in a peculiar dependence on the coupling constant. Secondly, contrary to the effective action in terms of the original gauge fields, the effective action now is proportional to an infraredly divergent factor, which makes the current stationary phase approximation much more accurate. And finally, the resulting behaviour of the calculated order parameter gives precisely the confinement mechanism Nikita Alexeevich has envisaged from the outset.

Of course, there always remains the big question how the renomalisation procedure in the next loop approximation would affect the picture. Nevertheless, infared and ultraviolet divergences do combine nicely together to produce a finite ratio of the string tension to the squared transition temperature. This, along with some other indications, led us to believe that the confinement is mostly infra–red dominated and that what we have obtained from a kind of mean–field is essentially correct.

This story, sadly, does not have an ending, but perhaps some day somebody would be in the position to verify this conjecture and to bring the confinement quest to a logical conclusion which Nikita Alexeevich was so desperately seeking in the last days of his tragically short life.

2 Theoretical approach

2.1 Fock–Schwinger gauge

The Fock–Schwinger gauge condition [1] is defined for the system in a spherical domain as,

$$\mathbf{A}_{\|} = \hat{\mathbf{x}}\,\mathbf{A}(x\hat{\mathbf{x}}) = 0, \qquad \hat{\mathbf{x}} \equiv \mathbf{x}/x, \quad x = |\mathbf{x}|. \tag{1}$$

Remarkably, here the Gauss law constraint can be resolved explicitly,

$$E_{\|}(\mathbf{x}) = E_{\|}(\mathbf{A}_{\perp}, \mathbf{E}_{\perp}) = -\frac{1}{x^2}\int_0^x y^2 dy\, \boldsymbol{\nabla}\mathbf{E}_{\perp}(y\hat{\mathbf{x}}), \tag{2}$$

where the right–hand side depends only on the physical transversal variables. The resulting Hamiltonian of the Yang–Mills theory [3] turns out to be only quartic in the dynamic variables [4, 5],

$$H_V = \frac{1}{2}\int_{\mathbf{x}\in V} d\mathbf{x}\, \left(\mathbf{E}_{\perp}^2 + E_{\|}^2 + \frac{1}{2}F_{ij}^2\right), \tag{3}$$

where the gauge strengths F_{ij} are only dependent on $\mathbf{A}_{\perp}$.

2.2 Confinement criterion

It is normally assumed that the dependence of the partition function on the boundary conditions imposed on all fields disappears in the thermodynamic limit. This is not obvious for the longitudinal component of the electric field. Really, according to Eq. (2) due to the non–locality fixing it to be some function on the boundary imposes a constraint on the values of the volume dynamic variables. Therefore, one is interested to study how the partition function in the infinite system,

$$Z_\beta[\chi] = \lim_{V\to\infty} \mathrm{Tr}\left(\exp(-\beta H_V)\,\delta(R^2 E_\parallel|_{\partial V} - \chi(\hat{\mathbf{x}}))\right) \tag{4}$$

depends on the Dirichlet boundary condition given by the function $\chi(\hat{\mathbf{x}})$.

The *confinement criterion* proposed by N.A. Sveshnikov is as follows. At high temperatures $T > T_c$, in the deconfinement phase, Z_β should not depend on χ in the thermodynamic limit. However, if there is a confinement–deconfinement phase transition, below the critical temperature the dependence should become very pronounced,

$$Z_\beta \sim \delta(\chi(\hat{\mathbf{x}})). \tag{5}$$

In other words, all non–vanishing values of χ become thermodynamically unfavorable. Since $E_\parallel$ is the generator of the large (non–vanishing at infinity) gauge transformations the latter condition is exactly equivalent to the singletness of the partition function,

$$Z_\beta[0] = \lim_{V\to\infty} \mathrm{Tr}\left(\exp(-\beta H_V)\,\mathcal{P}_s\right), \tag{6}$$

where $\mathcal{P}_s$ is the singlet projector of such gauge transformations. Quite curiously, this in turn would yield the area law for the Wilson loop [6].

Now, one only has to study the dependence $Z_\beta[\chi]$ to see if such a criterion holds.

2.3 Partition function vs the boundary condition

In Ref. [5] we have shown that the partition function can be identically rewritten in terms of the two collective variables,

$$Z_\beta[\chi] = \int \mathcal{D}\sigma\,\mathcal{D}\nu\,\exp\left(-W[\sigma,\nu] + i\int_{\partial\Lambda} dt\,d\hat{\mathbf{x}}\,\sigma\chi\right)\delta(R^2\sigma' - i\chi), \tag{7}$$

which have the meaning of the longitudinal components of an integral of the electric field and of the magnetic field.

The resulting effective action seems quite complicated,

$$\begin{aligned} 2W[\sigma,\nu] &= \nu\bullet\nu + \partial\sigma\bullet\partial\sigma + K_-\bullet C_+^{-1}\bullet K_+ \\ &+ K_+\bullet C_-^{-1}\bullet K_- + \mathrm{tr}\log C_+C_-\,, \\ C_\pm &= -\Delta_x - \nabla_t^2 \pm D\,, \quad K_\pm = \partial_\pm\nu \pm \nabla_t\partial_\pm\sigma\,, \\ \nabla_t^{ab} &= \delta^{ab}\partial_t - gt^{abc}\,\sigma^c\,, \quad D^{ab} = gt^{abc}\,\nu^c\,, \end{aligned} \tag{8}$$

where the projected derivatives are defined by,

$$\partial_\pm^i = \Pi_\pm^{ij}\,\partial_j\,, \quad \Pi_\pm^{ij} = \frac{1}{2}(\delta^{ij} - \hat{x}^i\hat{x}^j \pm i\epsilon^{ijk}\hat{x}^k)\,. \tag{9}$$

2.4 The mean–field approximation

In the mean–field approximation we need to find the stationary points of the effective action:

$$\frac{\delta W}{\delta\sigma} = 0, \qquad \frac{\delta W}{\delta\nu} = 0. \tag{10}$$

Note that there are residual gauge transformations at infinity, $A \to U^{-1}AU + g^{-1}U^{-1}\partial U$ with $U = U(\hat{\mathbf{x}})$. Because we know that there is no spontaneous breaking of the colour symmetry the mean–field partition function has to be integrated over all equivalent solutions. For SU(2) integration over an orbit of the residual gauge group yields,

$$Z_\beta[\chi] \sim \prod_{\hat{\mathbf{x}}} \frac{\sin|\chi|\sigma\beta\Delta}{|\chi|\sigma\beta\Delta} \sim \exp\left(-\frac{\Delta\beta^2R^2}{6}\int_{\partial V} d\hat{\mathbf{x}}\,\sigma^2\,\chi_a^2(\hat{\mathbf{x}})\right). \tag{11}$$

In Refs. [4, 5] we have found that only below the critical temperature the solution is nonzero,

$$\sigma = \frac{1}{\sqrt{\Delta}} \sqrt{\frac{3}{4\pi}} \frac{a_c}{R}, \qquad a_c \simeq 1.22. \tag{12}$$

This results in the following dependence of the partition function on the boundary condition,

$$Z_\beta[\chi] \sim \exp\left(-\frac{\beta^2 a_c^2}{8\pi} \int_{\partial V} d\hat{\mathbf{x}}\, \chi_a^2(\hat{\mathbf{x}}) \right) \exp(-W[\sigma]), \tag{13}$$

$$W[\sigma] \sim \frac{V}{g^2} w[\sigma], \tag{14}$$

where $w[\sigma]$ is a regular function. Thus, below the critical temperature $Z_\beta[\chi] < Z_\beta[0]$, so only the vanishing boundary condition is thermodynamically realisable.

Having derived the partition function we can find the Gibbs average of any observable A,

$$< A >= \begin{cases} \langle A \rangle_0, & T < T_c, \\ \int \mathcal{D}\chi(\hat{\mathbf{x}})\, \langle A \rangle_\chi, & T > T_c, \end{cases} \tag{15}$$

The former average in the confinement phase then contains the singlet projector $\mathcal{P}_s$ of the big gauge transformations at infinity,

$$\langle A \rangle_0 = \lim_{R \to \infty} \frac{1}{Z_R[0]} \mathrm{Tr}\, (e^{-\beta H_R}\, \mathcal{P}_s\, A)\,. \tag{16}$$

3 Conclusion

In conclusion I would like to relate the work described here to the theoretical program of study of confinement that N.A. Sveshnikov had.

The first step of his program was to find the confinement criterion, confirm it by direct calculation of the partition function and to understand its meaning. Probably, this may be viewed as, at least, mainly accomplished.

The next step of the program is to study the consequences of confinement. There is a number of phenomenological expectations even for simple observables such as condensates, asymptotics of the propogators and others, that should be significantly affected by confinement. For that one should take into account the singlet projector during their calculation. Such calculations are not simple in the 3-d Fock–Schwinger gauge though. The only straightforward successful calculation of observables so far has been done for the Wilson loop in Ref. [6].

I believe that Nikita Alexeevich had a good point in expecting that a further progress could be made using the Bogoliubov approach, provided the latter was successfully constructed. There are other alternatives, but a number of technical difficulties appears in doing non–perturbative derivations.

The ultimate goal of the program would be to construct the phenomenology of strong interactions starting from the fundamental principles of the Yang–Mills theory and equipped with the proper understanding of confinement phenomenon achieved as the result of the previous steps. This still remains a grand challenge for high energy theorists for some years to come.

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