Cosmological solutions with gauge and spinor fields

Gerd Rudolph and Torsten Tok
Institute of Theoretical Physics, University Leipzig, 04109 Leipzig, Augustusplatz 10
Germany
Igor Volobuev
Institute of Nuclear Physics, Moscow State University, 119899 Moscow
Russia

Abstract

We present exact solutions in Einstein-Yang-Mills-Dirac theories with gauge groups $SU(2)$ and $SU(4)$ in Robertson-Walker space-time $\mathbb{R} \times S^3$, which are symmetric under the action of the group $SO(4)$ of spacial rotations. We interpret these solutions as cosmological solutions for the radiation dominated universe, in which an exchange of energy between the Dirac and Yang-Mills fields takes place.

Our approach is based on the dimensional reduction method for gauge and gravitational fields and relates symmetric solutions in EYMD theory to certain solutions of an effective dynamic system. The explicit form of the solution for spinor field shows that its energy changes the sign during the evolution of the Yang-Mills field from one vacuum to the other, which can be considered as production or annihilation of fermions.

1 Introduction

In accordance with the standard cosmological model, the evolution of the Universe is described by homogeneous and isotropic solutions in Einstein gravity coupled to energy-momentum tensors of the matter, which have different structure for different epochs of the evolution. For example, the inflationary phase is described by the energy-momentum tensor of a scalar inflaton field, travelling from the symmetric vacuum to the Higgs vacuum and providing the energy necessary for the inflation, whereas the later epochs are described by phenomenological energy-momentum tensors, first by that of the radiation and then by that of the dust.

This description cannot be considered as quite satisfactory, because it leaves gaps between the phases of the evolution. A more consistent approach would be the one, which provides the necessary structure of the energy-momentum tensor for each epoch and describes the transition from one epoch to another. Such an approach could be based on the fundamental Lagrangians of the particle physics and, in principle, could relate the phase transitions in unification models with those in the Universe.

As a first step in the implementation of this programme, one looks for exact solutions in gravity coupled to classical fields of different types. In fact, such solutions have always attracted much attention. In particular, the last few years witnessed a great interest in solutions to EYM systems. There were found both numerical [1] and exact solutions with $SO(3)$ and $SO(4)$ groups of spacial symmetry [2, 3]. The exact $SO(3)$-symmetric solutions turn out to be static and singular and are a generalization of Reissner-Nordström solutions to non-abelian gauge theories. The $SO(4)$-symmetric solutions correspond to the Robertson-Walker ansatz for the metric and are interpreted either as wormhole solutions in the euclidean domain [4] or as cosmological solutions for the radiation dominated universe [5].

There were attempts to accommodate spinor fields in this picture as well, but they usually amounted to considering spinor fields in the EYM background. For example, in [6, 7] an anomalous fermion production in the EYM background was discussed.

On the other hand, the study of exact cosmological solutions in Einstein-Dirac systems also attracted much attention [8, 9]. It was found that the Robertson-Walker ansatz in such systems leads to the so-called "ghost solutions", for which the classical Dirac field has a vanishing energy-momentum tensor.

In the present paper we combine both approaches and consider a self-consistent EYMD system, in which we find exact solutions for the case of the gauge groups $SU(2)$ and $SU(4)$. The group $SU(4)$ is the simplest gauge group which gives qualitatively new results in comparison with the group $SU(2)$. Our solutions describe the radiation dominated universe, in which an exchange of energy between the YM and Dirac fields takes place. We believe that they can be of interest for treating the dynamics of the early

Universe. We also propose a physical interpretation of the classical Dirac field, which solves the problem of the "ghost solutions" in ED systems.

To find these solutions we employ the dimensional reduction method for gravitational, gauge and spinor fields [10, 11], which enables us to relate symmetric solutions in EYMD system to certain solutions of an effective dynamic system and essentially simplifies the problem of finding symmetric solutions.

Our exposition is organized as follows. In section 2 we present the main facts about invariant metrics and symmetric gauge and spinor fields and obtain the reduced effective action for our symmetric EYMD system. In section 3 we derive the corresponding equations of motion and find formal solutions to these equations. Then in section 4 we discuss physical interpretation of our classical Dirac field and, on the basis of this interpretation, select physically adequate solutions.

2 The effective action

We consider an $SO(4)$-symmetric Einstein-Yang-Mills-Dirac system with the standard action in space-time $M = \mathbb{R} \times S^3$:

$$S = S_E + S_{YM} + S_D \,, \tag{1}$$

where

$$\begin{aligned} S_E &= \frac{1}{16\pi\kappa}\int_M (\mathrm{R} - \Lambda)\,\mathrm{d}v\,, & (2)\\ S_{YM} &= \frac{1}{8g^2}\int_M \mathrm{tr}(\mathrm{F}_{\alpha\beta}\mathrm{F}^{\alpha\beta})\,\mathrm{d}v\,, & (3)\\ S_D &= \int_M \frac{i}{2}\bar\psi\gamma^\alpha E_\alpha{}^\mu\left(\frac{\partial}{\partial x^\mu} - \frac{1}{8}\omega_{\mu\alpha\beta}[\gamma^\alpha,\gamma^\beta] + \delta'(\mathrm{A}_\mu)\right)\psi\,\mathrm{d}v + h.c.\,. & (4)\end{aligned}$$

Here Λ denotes the cosmological constant, R is the scalar curvature and $dv = \sqrt{|\det(\mathbf{g})|}\mathrm{d}^4x$ is the canonical volume form corresponding to a metric $\mathbf{g}$ on $M = \mathbb{R} \times S^3$. F and A are the Yang-Mills field strength and potential respectively, and the trace is taken in the adjoint representation of the Lie algebra of the gauge group. In the Dirac action, ψ denotes the spinor field, $\{E_\alpha = E_\alpha{}^\mu\partial_\mu\}$ is an orthonormal frame on $M = \mathbb{R} \times S^3$, $\{\gamma^\alpha\}$ are the gamma matrices and δ is the representation of the gauge group in the spinor space.

We identify S^3 with the group manifold of $SU(2)$. Then the action σ of the symmetry group $K = SO(4) \equiv (SU(2) \times SU(2))/\{\mathbf{1}, -\mathbf{1}\}$ on $SU(2)$ is given by

$$\sigma((k_1,k_2),x) = k_1 x k_2{}^{-1}\,, \quad (k_1,k_2) \in SU(2) \times SU(2)\,,\, x \in SU(2)\,. \tag{5}$$

The isotropy subgroup H at $x = \mathbf{1}_{SU(2)}$ is isomorphic to $SU(2)$ and is given by

$$H = \{(k,k) \in K\,, \quad k \in SU(2)\}\,. \tag{6}$$

Further, we have the reductive decomposition $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{M}$ of the Lie algebra $\mathfrak{K}$ of K, where

$$\begin{aligned} \mathfrak{H} &= \{(X,X)\,; \quad X \in su(2)\}\,, & (7)\\ \mathfrak{M} &= \{(X,-X)\,; \quad X \in su(2)\}\,. & (8)\end{aligned}$$

Obviously $S^3 = (SU(2) \times SU(2))/H$ is a symmetric space, and $SO(4)$-invariance is equivalent to invariance under $SU(2) \times SU(2)$. The isotropy representation of $H = SU(2)$ in $\mathfrak{M}$ is $\underline{3}$, i.e. it is equivalent to the adjoint representation.

Now we have to reduce the action of the original theory due to the $SO(4)$-symmetry. We begin with the action of the gravitational field.

The most general form of an $SO(4)$-invariant metric $\mathbf{g}$ on M is

$$\mathbf{g} = -N^2\mathrm{d}t^2 + a^2\mathrm{d}\Omega^2_{S^3}\,, \tag{9}$$

where $\mathrm{d}\Omega^2_{S^3}$ is the standard line element on a 3-sphere of radius 2 corresponding to the Killing metric on $SU(2) \equiv S^3$ multiplied by $-\frac{1}{2}$. For the sake of the future convenience we assume N and a to have the dimension of length and t and $\mathrm{d}\Omega^2_{S^3}$ to be dimensionless.

An orthonormal coframe $\{\theta^\mu\}$ on M is given by

$$\theta^0 = N\mathrm{d}t\,, \quad \theta^i = a\vartheta^i\,,\, i = 1,2,3\,, \tag{10}$$

where $\vartheta^i\sigma_i/(2i) \equiv \vartheta$ is the canonical left invariant 1-form on $SU(2)$, σ_i being the Pauli matrices. In what follows we put $\tau_k = \sigma_k/(2i)$ and denote by $\eta_{\alpha\beta}$ the Minkowski metric $\eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$. It is a matter of simple calculations to find the components of the spin connection on M:

$$\begin{aligned} \omega_{0i} &= -\omega_{i0} = -\,\dot{a}\,\frac{1}{aN}\delta_{il}\theta^l\,, \\ \omega_{ik} &= -\frac{1}{2a}\varepsilon_{ikl}\theta^l\,, \quad i = 1,2,3\,. \end{aligned} \tag{11}$$

Using the standard formulae for the curvature, substituting it into (2) and integrating over S^3, one easily finds the reduced gravitational action

$$S_E = \frac{16\pi^2}{16\pi\kappa}\int_{\mathbb{R}} a^3N\left(\frac{3}{2a^2} + 6\left(\dot{a}\,\frac{1}{aN}\right)^2 - \dot{a}\dot{N}\,\frac{6}{aN^3} + \ddot{a}\,\frac{6}{aN^2} - \Lambda\right)\mathrm{d}t\,, \tag{12}$$

where $16\pi^2$ is the volume of $SU(2) = S^3$ with the standard metric $\mathrm{d}\Omega^2_{S^3}$. Omitting a complete divergence, we get the effective action

$$S_E = \frac{16\pi^2}{16\pi\kappa}\int_{\mathbb{R}}\left(\frac{3}{2}aN - \Lambda a^3N - 6\frac{1}{N}a\dot{a}^2\right)\mathrm{d}t\,, \tag{13}$$

which we consider as the reduced action of the gravitational field.

Next we turn to the gauge field action (3). An $SO(4)$-symmetric gauge potential A on $\mathbb{R}\times S^3$ is in one-to-one correspondence with a triplet $\{\tau\,, A\,, \Phi\}$, where τ is a homomorphism from the isotropy group H into the gauge group G

$$\tau : H \to G\,, \tag{14}$$

A is a gauge potential on $\mathbb{R}$ with values in the centralizer $\mathfrak{C}_{\mathfrak{G}}(\tau'(\mathfrak{H}))$ of $\tau'(\mathfrak{H})$ in $\mathfrak{G} = Lie(G)$ and Φ is a linear mapping

$$\Phi : \mathbb{R} \to \mathfrak{M}^* \otimes \mathfrak{G} \tag{15}$$

satisfying

$$\Phi \circ \mathrm{A}d(h) = \mathrm{A}d(\tau(h)) \circ \Phi\,, \quad \forall h \in H\,. \tag{16}$$

Here $\mathrm{A}d(h)$ is the restriction to H of the adjoint representation of K applied to $\mathfrak{M}$ and $\mathrm{A}d(\tau(h))$ is the restriction to $\tau(H)$ of the adjoint representation of G in $\mathfrak{G}$. In fact, equation (16) means that the mapping Φ intertwines these representations.

As we have already mentioned, we will consider gauge groups $SU(2)$ and $SU(4)$. In the first case the centralizer is trivial, and there is no reduced gauge potential A, whereas in the second case there can be a nontrivial centralizer. Therefore, the case of the group $SU(4)$ is more general, and we will carry out all the calculations for this case and then explain the difference from the case of $SU(2)$.

Thus, we take the gauge group $G = SU(4)$ and define the homomorphism $\tau : H \to G$ by the decomposition of the fundamental representation $\underline{4}$ of SU(4) :

$$\underline{4} \to (\underline{2}, \underline{2})\,, \tag{17}$$

i.e. we represent the space $\mathbb{C}^4$ as the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$ and let the fundamental representation of $\tau(H)$ act on the first factor and the fundamental representation of the centralizer $C_G(\tau(H)) = SU(2)$ act on the second factor.

It is known that the adjoint representation of $sl(n)$ can be expressed in terms of its fundamental representation by [12]

$$\mathrm{ad}\, sl(n) = \underline{n}^*\tilde{\otimes}\,\underline{n}\,, \tag{18}$$

where tilde means dropping a one-dimensional trivial representation and $\underline{n}^*$ denotes the contragradient representation, $t(x)^* = -t(x)^T$. Therefore, we get

$$\mathrm{ad}\, su(4) \to (\underline{2},\underline{2})^* \tilde{\otimes} (\underline{2},\underline{2}) = (\underline{3},\underline{1}) \oplus (\underline{3},\underline{3}) \oplus (\underline{1},\underline{3})\,. \tag{19}$$

A basis in $su(4)$ adapted to this decomposition is

$$H_k = \tau_k \otimes \mathbf{1} = \begin{pmatrix} \tau_k & 0 \\ 0 & \tau_k \end{pmatrix}, \tag{20}$$

$$P_{kA} = \tau_k \otimes \sigma_A\,, \tag{21}$$

$$H_A = \mathbf{1} \otimes \tau_A\,, \tag{22}$$

where equation (20) explains our rule for evaluating the tensor product of matrices. It is easy to calculate the commutators of these generators, for instance

$$[P_{kA}, P_{lB}] = \varepsilon_{klm} H_m \delta_{AB} + \varepsilon_{ABC} H_C \delta_{kl}\,. \tag{23}$$

The constraint (16) means that the mapping Φ is an intertwining operator, which intertwines the isotropy representation $\underline{3}$ of $H = SU(2)$ in $\mathfrak{M}$ with the representation (19). We introduce basic intertwining operators from $\mathfrak{M}$ into $\mathfrak{G}$ defined by the relations

$$I((\tau_k, -\tau_k)) = H_k\,, \tag{24}$$

$$I_A((\tau_k, -\tau_k)) = P_{kA}\,. \tag{25}$$

Then for any $X = (X^k \tau_k) \in su(2)$ we have

$$\Phi((X,-X)) = \xi I(X,-X) + \xi^A I_A(X,-X) \tag{26}$$

$$= X^k(\xi H_k + \xi^A P_{kA}) = X \otimes (\xi \mathbf{1} + \xi^A \sigma_A)\,, \tag{27}$$

where ξ and ξ^A, $A = 1,2,3$, are real valued functions on $\mathbb{R}$. In what follows we denote the vector (ξ^1, ξ^2, ξ^3) by $\vec{\xi}$ and put $\hat{\xi} = \xi\mathbf{1} + \xi^A \sigma_A$ and $\tilde{\xi} = \xi^A \sigma_A$. The centralizer $\mathfrak{C}_{su(4)}(\tau'(su(2)))$ of $\tau'(su(2))$ in $\mathfrak{G} = su(4)$ is $su(2)$ and is spanned by the Lie algebra elements $H_A, A = 1,2,3$. The matrix $\hat{\xi}$ is in the representation $(\underline{1} + \underline{3})$ of the centralizer $\mathfrak{C}$ and $i\hat{\xi} \in u(2)$.

The symmetric gauge potential A on $\mathbb{R} \times S^3$ can be easily expressed in terms of the matrix $\hat{\xi}$ and the canonical left invariant 1-form $\vartheta = \vartheta^i \tau_i$ on $SU(2) \equiv S^3$:

$$\begin{aligned} \mathrm{A} &= \frac{\xi+1}{2} H_i \vartheta^i + \frac{1}{2}\xi^A P_{kA}\vartheta^k + \mathrm{A}_0^A H_A \mathrm{d}t \\ &= \frac{1}{2}\vartheta \otimes \left(\hat{\xi} + \mathbf{1}\right) + \mathbf{1} \otimes \mathrm{A}_0 \mathrm{d}t\,. \end{aligned} \tag{28}$$

Here A_0 is a function on $\mathbb{R}$ with values in $su(2)$, i.e. $\mathrm{A}_0 = \mathrm{A}_0^A \tau_A$. The term $\mathbf{1} \otimes \mathrm{A}_0 \mathrm{d}t$ is the reduced gauge potential on $\mathbb{R}$, which can be gauged out, but we keep it for the moment, because it is necessary for deriving the equations of motion. It is not difficult to calculate the corresponding field strength F and to obtain the reduced Yang-Mills action, which takes the simple form

$$\begin{aligned} S_{YM} &= \frac{16\pi^2}{8g^2}\int_{\mathbb{R}} 24\Bigg(\frac{a}{2N}\,\dot{\xi}^2 + \frac{a}{2N}(\dot{\vec{\xi}} + \vec{\mathrm{A}}_0 \times \vec{\xi})^2 \\ &\quad -\frac{N}{8a}\left((\xi^2 + (\vec{\xi})^2 - 1)^2 + 4\xi^2(\vec{\xi})^2\right)\Bigg)\mathrm{d}t\,, \end{aligned} \tag{29}$$

where $\vec{\mathrm{A}}_0$ is the vector $(\mathrm{A}_0^1, \mathrm{A}_0^2, \mathrm{A}_0^3)$ and $\vec{\mathrm{A}}_0 \times \vec{\xi}$ is the vector product of $\vec{\mathrm{A}}_0$ and $\vec{\xi}$.

If the gauge group is $G = SU(2)$, the unique nontrivial homomorphism $\tau : H \to G$ can be defined by the identity mapping, i.e.

$$\tau((k,k)) = k\,, \quad k \in SU(2)\,. \tag{30}$$

In this case the centralizer $\mathfrak{C}_{\mathfrak{G}}(\tau'(\mathfrak{H}))$ is trivial, and the intertwining operator is

$$\Phi((X,-X)) = \xi(t)X\,, \quad X \in su(2)\,, \xi(t) \in \mathbb{R}\,. \tag{31}$$

It is clear that the gauge potential A on $\mathbb{R}\times S^3$ still has the form (28) with $\vec{\xi}=0, \mathrm{A}_0=0$. The reduced Yang-Mills action is also given by (29), provided one puts $\vec{\xi}=0$ and rescales the coupling constant $g\to 2g$.

Next we have to reduce the action (4) for symmetric spinor field. We choose δ to be the fundamental representation, i.e. we can write the spinor ψ as a 4×4 resp. 4×2 matrix on which an element g of $SU(4)$ resp. $SU(2)$ acts via right multiplication by g^{-1}.

In accordance with our choice of the metric signature (9), we have

$$\{\gamma^\mu,\gamma^\nu\} = -2\eta^{\mu\nu}\,, \quad \eta^{\mu\nu} = \mathrm{diag}(-1,1,1,1)\,, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,, \tag{32}$$

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, i=1,2,3\,, \gamma^5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{33}$$

The bispinor representation $\Delta(s)$ is defined by

$$\Delta(s)^{-1}\gamma^\mu\Delta(s) = \Lambda(s)^\mu{}_\rho\gamma^\rho\,, \tag{34}$$

where s is an element of the group $Spin(1,3)$ and Λ is the covering homomorphism from $Spin(1,3)$ onto $SO(1,3)$. On Lie algebra level we get

$$[\Delta'(A),\gamma^\mu] = -A^\mu{}_\nu\gamma^\nu\,, \quad \Delta'(A) = -\frac{1}{8}A^\mu{}_\nu[\gamma_\mu,\gamma^\nu]\,, \tag{35}$$

where $A_{\mu\nu}=-A_{\nu\mu}$ is an element of $so(1,3)\equiv spin(1,3)\equiv sl(2,\mathbb{C})$.

An $SO(4)$-symmetric spinor field ψ on $\mathbb{R}\times S^3$ is in one-to-one correspondence with a matrix valued function ρ on $\mathbb{R}$, which satisfies the condition

$$(\delta'(\tau'(h)) + \Delta'(\lambda'(h)))\,\rho = 0 \qquad, \forall h\in\mathfrak{H}\,. \tag{36}$$

Here $\lambda' : \mathfrak{H}\to so(1,3)$ is the homomorphism induced by the isotropy representation, which can be calculated explicitly: $\lambda'(\tau_a)^b{}_c = -\varepsilon_a{}^b{}_c$. Therefore, if the gauge group G is $SU(4)$ equation (36) reads

$$\left(\frac{1}{4}\gamma^i\gamma^j\varepsilon_{ijk} + \delta'(H_k)\right)\rho = ((\tau_k\otimes\mathbf{1})\,\rho - \rho\,(\tau_k\otimes\mathbf{1})) = 0\,. \tag{37}$$

The general solution of this constraint equation is

$$\rho = \begin{pmatrix} u_1\mathbf{1} & u_2\mathbf{1} \\ v_1\mathbf{1} & v_2\mathbf{1} \end{pmatrix} = \mathbf{1}\otimes\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}, \quad u_1,u_2,v_1,v_2\in C^\infty(\mathbb{R})\,, \tag{38}$$

i.e. a symmetric spinor on $\mathbb{R}\times S^3$ is parameterized by two complex doublets $u=(u_1,u_2)^T$ and $v=(v_1,v_2)^T$, one for each chirality. We see from (38) that the reduced gauge group $C=SU(2)$ acts on both doublets by the fundamental representation. Taking into account equations (11) and (28), it is a matter of simple calculations to get the reduced action

$$\begin{aligned} S_D = 16\pi^2\int_{\mathbb{R}} \quad a^3N \quad & \Big(\frac{i}{N}(\bar{u}\,\dot{u} - \dot{\bar{u}}\,u + \bar{v}\,\dot{v} - \dot{\bar{v}}\,v) \\ & + \frac{1}{N}A_0^B(\bar{u}\sigma_B u + \bar{v}\sigma_B v) - \frac{3}{2a}(\bar{u}\hat{\xi}u - \bar{v}\hat{\xi}v)\Big)\mathrm{d}t\,. \end{aligned} \tag{39}$$

If the gauge group G is $SU(2)$, equation (36) reads

$$((\tau_k\otimes\mathbf{1})\,\rho - \rho\,\tau_k) \quad = \quad 0\,, \tag{40}$$

and we get

$$\rho = \begin{pmatrix} u\mathbf{1} \\ v\mathbf{1} \end{pmatrix} = \mathbf{1} \otimes \begin{pmatrix} u \\ v \end{pmatrix}, \quad u, v \in C^\infty(\mathbb{R}), \tag{41}$$

i.e. a symmetric spinor on $\mathbb{R} \times S^3$ for $G = SU(2)$ depends on two arbitrary complex functions u and v, one for each chirality. The reduced action has the same form (39), if we put there $\vec{\xi} = 0, \mathrm{A}_0 = 0$.

Now we can write down the reduced action of the coupled EYMD system. In the case of the gauge group $SU(4)$ it has the form

$$\begin{aligned} S &= S_E + S_{YM} + S_D \\ &= 16\pi^2 \int_{\mathbb{R}} \Bigg\{ \frac{1}{16\pi\kappa} \left(\frac{3}{2} aN - \Lambda a^3 N - 6 \frac{a}{N} \dot{a}^2 \right) \\ &\quad + \frac{24}{8g^2} \Big(\frac{a}{2N} \dot{\xi}^2 + \frac{a}{2N} (\dot{\vec{\xi}} + \vec{A}_0 \times \vec{\xi})^2 - \frac{N}{8a} ((\xi^2 + (\vec{\xi})^2 - 1)^2 + 4\xi^2 (\vec{\xi})^2) \Big) \\ &\quad + \Big(i a^3 (\bar{u}\,\dot{u} - \dot{\bar{u}}\,u + \bar{v}\,\dot{v} - \dot{\bar{v}}\,v) \\ &\quad + a^3 A_0^B (\bar{u}\sigma_B u + \bar{v}\sigma_B v) - \frac{3a^2 N}{2} (\bar{u}\hat{\xi} u - \bar{v}\hat{\xi} v) \Big) \Bigg\} \mathrm{d}t\,. \end{aligned} \tag{42}$$

If we choose the gauge group to be $SU(2)$, we have to put $\vec{\xi} = 0$, $\mathrm{A}_0 = 0$ in this action, to rescale the coupling constant $g \to 2g$, and to take into account that the variables u and v are no longer isospinors, but ordinary functions.

3 The field equations and solutions

Variation of this action with respect to a, N, ξ, $\vec{\xi}$, u, v and $\bar{u}$, $\bar{v}$ gives us the field equations. When taken in the special gauge $\mathrm{A}_0 = 0$, $a = N$ (the latter condition means that t is now the conformal time), they have the form

$$\begin{aligned} a\frac{\delta}{\delta a} S &= \frac{1}{16\pi\kappa} \left(\frac{3}{2} a^2 - 3\Lambda a^4 - 6\dot{a}^2 + 12 a\ddot{a} \right) \\ &\quad + \frac{3}{2g^2} \left(\dot{\xi}^2 + (\dot{\vec{\xi}})^2 + \frac{1}{4} \left((\xi^2 + (\vec{\xi})^2 - 1)^2 + 4\xi^2 (\vec{\xi})^2 \right) \right) \\ &\quad + 3a^3 \left(i(\bar{u}\,\dot{u} - \dot{\bar{u}}\,u + \bar{v}\,\dot{v} - \dot{\bar{v}}\,v) - (\bar{u}\hat{\xi} u - \bar{v}\hat{\xi} v) \right) = 0\,, \end{aligned} \tag{43}$$

$$\begin{aligned} a\frac{\delta}{\delta N} S &= \frac{1}{16\pi\kappa} \left(\frac{3}{2} a^2 - \Lambda a^4 + 6\dot{a}^2 \right) \\ &\quad - \frac{3}{2g^2} \left(\dot{\xi}^2 + (\dot{\vec{\xi}})^2 + \frac{1}{4} \left((\xi^2 + (\vec{\xi})^2 - 1)^2 + 4\xi^2 (\vec{\xi})^2 \right) \right) \\ &\quad - \frac{3}{2} a^3 (\bar{u}\hat{\xi} u - \bar{v}\hat{\xi} v) = 0\,, \end{aligned} \tag{44}$$

$$\begin{aligned} \frac{\delta}{\delta \xi} S &= -\frac{3}{2g^2} \left(2\ddot{\xi} + \left(\xi(\xi^2 + (\vec{\xi})^2 - 1) + 2\xi (\vec{\xi})^2 \right) \right) \\ &\quad - \frac{3}{2} a^3 (\bar{u} u - \bar{v} v) = 0\,, \end{aligned} \tag{45}$$

$$\begin{aligned} \frac{\delta}{\delta \xi^A} S &= -\frac{3}{g^2} \left(\ddot{\xi}^A + \frac{1}{2} \left(\xi^A (\xi^2 + (\vec{\xi})^2 - 1) + 2\xi^2 \xi^A \right) \right) \\ &\quad - \frac{3}{2} a^3 (\bar{u}\sigma^A u - \bar{v}\sigma^A v) = 0\,, \end{aligned} \tag{46}$$

$$
\begin{aligned}
\frac{1}{2}a^{-\frac{3}{2}}\frac{\delta}{\delta\bar{u}}S &= i\frac{d}{dt}a^{\frac{3}{2}}u - \frac{3}{4}\hat{\xi}a^{\frac{3}{2}}u = 0\,, \\
\frac{1}{2}a^{-\frac{3}{2}}\frac{\delta}{\delta u}S &= i\frac{d}{dt}a^{\frac{3}{2}}\bar{u} + \frac{3}{4}a^{\frac{3}{2}}\bar{u}\hat{\xi} = 0\,,
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
\frac{1}{2}a^{-\frac{3}{2}}\frac{\delta}{\delta\bar{v}}S &= i\frac{d}{dt}a^{\frac{3}{2}}v + \frac{3}{4}\hat{\xi}a^{\frac{3}{2}}v = 0\,, \\
\frac{1}{2}a^{-\frac{3}{2}}\frac{\delta}{\delta v}S &= i\frac{d}{dt}a^{\frac{3}{2}}\bar{v} - \frac{3}{4}a^{\frac{3}{2}}\bar{v}\hat{\xi} = 0\,.
\end{aligned} \tag{48}
$$

Variation with respect to A_0 gives a constraint

$$
\frac{\delta S}{\delta \mathrm{A}_0^B} = \frac{3}{g^2}\left(\varepsilon_{BCD}\xi^C\dot{\xi}^D\right) + a^3(\bar{u}\sigma_B u + \bar{v}\sigma_B v) = 0\,, \tag{49}
$$

which means that the total isospin of the gauge and the spinor fields equals zero.

Now we will show that it is possible to find exact solutions to this system of equations. We begin with the simpler case of the gauge group $SU(2)$. There is no constraint (49) in this case, and we also have to drop the equation (46), to put $\vec{\xi} = 0$ in the others and to rescale $g \to 2g$.

The Dirac equations (47) and (48) give

$$
\bar{u}u = \frac{C_u}{a^3}\,, \quad \bar{v}v = \frac{C_v}{a^3} \tag{50}
$$

and

$$
\bar{u}\,\dot{u} - \dot{\bar{u}}\,u = -\frac{3i}{2}\xi\bar{u}u\,, \tag{51}
$$

$$
\bar{v}\,\dot{v} - \dot{\bar{v}}\,v = \frac{3i}{2}\xi\bar{v}v\,, \tag{52}
$$

where C_u and C_v are arbitrary positive constants. We will discuss the meaning of these constants later, here we assume that they are proportional to the number of fermions with positive resp. negative chirality on S^3.

Now equations (43) and (44) simplify considerably. Their sum gives

$$
\frac{1}{16\pi\kappa}\left(3a^2 - 4\Lambda a^4 + 12a\ddot{a}\right) = 0\,. \tag{53}
$$

Multiplying this equation by $\dot{a}/a$ and integrating we see that

$$
\frac{3}{2}a^2 - \Lambda a^4 + 6\dot{a}^2 = E\,, \tag{54}
$$

where E is an arbitrary constant which has the meaning of the total energy of the system. Equation (54) is the standard Friedman equation for the radiation dominated universe and has a simple analogue in mechanics. We can consider a as the coordinate of a particle with unit mass and energy $E/12$ which moves in a potential

$$
W(a) = \frac{1}{8}a^2 - \frac{1}{12}\Lambda a^4\,. \tag{55}
$$

If $\Lambda < \Lambda_E = \frac{9}{16E}$, then the motion will be periodical. This means that our solution describes a universe which first expands and then contracts, where $a = 0$ corresponds to a singular metric in the beginning and the end. If $\Lambda \geq \Lambda_E$, then the solution can be either static or can describe an expanding universe.

Next we turn to the YM equation (45) (we recall that we have rescaled $g \to 2g$ in the case under consideration). Due to equation (50), it decouples from the equations for u and v:

$$
\frac{3}{8g^2}\left(2\ddot{\xi} + \left(\xi(\xi^2 - 1)\right)\right) + \frac{3}{2}(C_u - C_v) = 0\,. \tag{56}
$$

The first integral of this equation is

$$\frac{3}{8g^2}\left(\dot{\xi}^2+\frac{1}{4}(\xi^2-1)^2\right)+\frac{3}{2}\xi(C_u-C_v) \quad = \quad \frac{E}{16\pi\kappa}\,, \tag{57}$$

where the integration constant is due to equations (44) and (54).

Equation (57) also has an analogue in mechanics. A point particle with unit mass, energy $g^2E/(12\pi\kappa)$ and coordinate ξ moves in a double-well potential

$$V(\xi)=\frac{1}{8}(\xi^2-1)^2+2g^2\xi(C_u-C_v)\,. \tag{58}$$

We can interpret the first term in (57) as the energy of the Yang-Mills field and the second term as the energy of the Dirac field due to the interaction with the gauge field. The equation describes an exchange of energy of the two fields, the coupling between the gauge and the spinor field being proportional to the difference in the numbers of left and right handed fermions C_u-C_v. The exact solution of equation (57) is possible in terms of elliptic functions of the first kind [13].

It is not difficult to see that for sufficiently small $\mid 2g^2(C_u-C_v)\mid$ potential (58) has three extrema: the absolute minimum, a local minimum and a local maximum. We consider for instance the case, where $C_u-C_v<0$ and the energy of the system $g^2E/(12\pi\kappa)$ lies between the local minimum and the local maximum of the potential $V(\xi)$. Then the system will move between the turning points defined by the real zeros of the polynomial

$$\frac{g^2E}{12\pi\kappa}-V(\xi). \tag{59}$$

In the case under consideration we have four real zeros $d<c<b<a$, i.e.

$$\frac{1}{2}\dot{\xi}^2=\frac{g^2E}{12\pi\kappa}-V(\xi)=\frac{1}{8}(a-\xi)(b-\xi)(c-\xi)(\xi-d)\,. \tag{60}$$

Moreover, the form of the potential (58) stipulates that in the case $C_u-C_v<0$ we have $d<c<0$.

The equations of motion have two solutions with either $b\le\xi\le a$ or $d\le\xi\le c$. The solution for the latter case is given by the integral

$$t(\xi)=\int_d^{\xi}\frac{dx}{\sqrt{\frac{g^2E}{6\pi\kappa}-\frac{1}{4}(x^2-1)^2-4g^2x(C_u-C_v)}}\,, \tag{61}$$

which can be solved explicitly [13]

$$t(\xi)=\frac{2}{\sqrt{(a-c)(b-d)}}F\left(\arcsin\sqrt{\frac{(a-c)(\xi-d)}{(c-d)(a-\xi)}}\,,\,\sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}\right)\,, \tag{62}$$

where F is the elliptic integral of the first kind. This function can be inverted, and we can get $\xi(t)$ expressed in terms of the Jacobi elliptic function *sinus amplitudinis*. Solutions for other values of E and C_u-C_v can be written out in a similar way with the help of the formulae available in [13].

With a given solution ξ we can solve the Dirac equations (47) and (48)

$$u \quad = \quad \sqrt{\frac{C_u}{a^3}}\exp\left\{-i\int\frac{3}{4}\xi\mathrm{d}t\right\}, \tag{63}$$

$$v \quad = \quad \sqrt{\frac{C_v}{a^3}}\exp\left\{i\int\frac{3}{4}\xi\mathrm{d}t\right\}. \tag{64}$$

One easily checks that our solution fulfills the whole system of field equations.

Now we consider the case of gauge group $SU(4)$. The Dirac equations (47) and (48) again lead to (50), but instead of (51) and (52) we now have

$$\bar{u}\,\dot{u}-\dot{\bar{u}}\,u \quad = \quad -\frac{3i}{2}\bar{u}\hat{\xi}u\,, \tag{65}$$

$$\bar{v}\,\dot{v}-\dot{\bar{v}}\,v \quad = \quad \frac{3i}{2}\bar{v}\hat{\xi}v\,. \tag{66}$$

These equations are also sufficient to decouple the Friedman equation from the Yang-Mills-Dirac equations, and we get again the equation (54) for the scale factor a.

Now we have to solve the Yang-Mills equations (45) and (46). We start with the discussion of the constraint (49). In what follows we restrict ourselves to the case, where

$$\varepsilon_{BCD}\xi^C\dot{\xi}^D = 0 \quad \text{for} \quad B = 1,2,3\,, \tag{67}$$

i.e. the isospins of the gauge field and the Dirac field vanish separately. This equation means that the angular momentum of the motion in the $\vec{\xi}$-space equals zero, that is the motion goes along a straight line passing through the origin, and the vector $\vec{\xi}$ is always proportional to a fixed vector $\vec{\xi_0}$. We use the remaining gauge freedom to choose $\vec{\xi_0} = (0,0,1)$, i.e. $\tilde{\xi} = \zeta\sigma_3$, $\zeta \in \mathbb{R}$. In this gauge we obtain from the Yang-Mills equations (46)

$$\bar{u}\sigma_A u - \bar{v}\sigma_A v = 0 \quad \text{for} \quad A = 1,2\,. \tag{68}$$

Further we get from equations (49) and (67)

$$\bar{u}\sigma_B u + \bar{v}\sigma_B v = 0\,, \quad B = 1,2,3\,. \tag{69}$$

Hence, we have $\bar{u}\sigma_1 u = \bar{u}\sigma_2 u = 0$. These equations have two solutions:

$$\text{1st case:} \qquad u = a^{-3/2}\alpha_1 w_+\,, \quad v = a^{-3/2}\beta_1 w_- \tag{70}$$

$$\text{2nd case:} \qquad u = a^{-3/2}\alpha_2 w_-\,, \quad v = a^{-3/2}\beta_2 w_+\,, \tag{71}$$

$$\text{where} \qquad w_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; w_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{72}$$

and $\alpha_i, \beta_i \in \mathbb{C}$, $|\alpha_i| = |\beta_i|$, $i = 1,2$. In particular, we obtain $C_u = C_v = |\alpha_i|^2$ in the i'th case. If we put $\tilde{\xi} = \zeta\sigma_3 \equiv 0$, then the Yang-Mills equations (46) demand $u = v = 0$. Therefore, we consider only the nontrivial case, when $\tilde{\xi} \not\equiv 0$. Hence we have two Yang-Mills equations (45) and (46), which now take the form:

$$\frac{3}{2g^2}\left(2\ddot{\xi} + \left(\xi(\xi^2 + \zeta^2 - 1) + 2\xi\zeta^2\right)\right) = 0\,, \tag{73}$$

$$\frac{3}{2g^2}\left(2\ddot{\zeta} + \left(\zeta(\xi^2 + \zeta^2 - 1) + 2\xi^2\zeta\right)\right) + 3S = 0\,, \tag{74}$$

with $S := C_u = |\alpha_1|^2$ in the first and $S := -C_u = -|\alpha_2|^2$ in the second case.

To solve equations (73) and (74), we pass to new variables in accordance with

$$\xi = \frac{1}{2}(x+y)\,,$$
$$\zeta = \frac{1}{2}(x-y)\,. \tag{75}$$

It is easy to check that the equations for x and y decouple and take the form

$$\frac{3}{2g^2}\left(2\ddot{x} + x(x^2-1)\right) + 3S = 0\,, \tag{76}$$

$$\frac{3}{2g^2}\left(2\ddot{y} + y(y^2-1)\right) - 3S = 0\,. \tag{77}$$

The first integrals of these equations are

$$\frac{3}{2g^2}\left(\dot{x}^2 + \frac{1}{4}(x^2-1)^2\right) + 3Sx = \frac{E_1}{8\pi\kappa}\,, \tag{78}$$

$$\frac{3}{2g^2}\left(\dot{y}^2 + \frac{1}{4}(y^2-1)^2\right) - 3Sy = \frac{E_2}{8\pi\kappa}\,, \tag{79}$$

where due to (44) and (54) the constants E_1 and E_2 fulfill

$$E_1 + E_2 = E\,. \tag{80}$$

These equations can also be solved exactly in terms of Jacobi elliptic functions, but unlike the case of the group $SU(2)$, there will be two different periods of motion in x and y.

We would like to note here that equations of the type (73), (74) with $S = 0$ for the euclidean EYM system were first found in [4], but they were solved there only for the case either $\xi = 0$ or $\zeta = 0$. It is likely that the existence of two period can modify the equations for the energy quantization of the wormholes [4, 23].

Substituting the solutions for ξ and ζ into equation (28) we get

$$\mathrm{A} = \frac{1}{2}\theta \otimes \left(\hat{\xi} + \mathbf{1}\right) = \begin{pmatrix} \frac{1+x}{2}\,\theta & 0 \\ 0 & \frac{1+y}{2}\,\theta \end{pmatrix}, \tag{81}$$

i.e. the gauge potential A takes values only in the $su(2) \oplus su(2)$ subalgebra of $su(4)$, each $su(2)$ part of the gauge potential being coupled to only one of the spinor fields u resp. v.

If we have a solution to the system of equations (78) and (79) it is easy to integrate the Dirac equations (47) and (48). With given solutions ξ and ζ we obtain

$$u = w_+ \sqrt{\frac{C_u}{a^3}} \exp\left\{-i \int \frac{3}{4}\, x \,\mathrm{d}t\right\}, \tag{82}$$

$$v = w_- \sqrt{\frac{C_u}{a^3}} \exp\left\{i \int \frac{3}{4}\, y \,\mathrm{d}t\right\} \tag{83}$$

in the first and

$$u = w_- \sqrt{\frac{C_u}{a^3}} \exp\left\{-i \int \frac{3}{4}\, y \,\mathrm{d}t\right\}, \tag{84}$$

$$v = w_+ \sqrt{\frac{C_u}{a^3}} \exp\left\{i \int \frac{3}{4}\, x \,\mathrm{d}t\right\} \tag{85}$$

in the second case.

4 Discussion

In studying the self-consistent EYMD system we have found that the evolution of the metric decouples from the remaining system and is described by the Friedman equation for the radiation dominated universe. The same result for the case of EYM systems was obtained earlier in [4, 5], and solutions for the spinor field in this background were studied in [6]. Unlike the latter solutions, our solutions take into account the back reaction of the spinor field on the YM and gravitational fields.

At this point, we have to comment on the meaning of our classical spinor field. The problem of interpreting spinor fields in cosmology has been discussed for many years [8, 9], but so far no satisfactory interpretation has been found. It is clear that our solutions describe just one energy level of the Dirac field in the EYM background, which is the symmetric one and therefore exactly the lowest one. If we assume that the Dirac field is normalized to unit, the influence of this field on the EYM system is negligible. Therefore, we suggest that the Dirac field of our solutions is normalized arbitrarily and is, in fact, an effective field describing fermionic matter with $SO(4)$-invariant energy-momentum tensor. This assumptions seems to be reasonable in the cosmological setting, because the energy levels of the Dirac field in the EYM background after the inflation must be very dense, and replacing the contribution of fermions on the excited levels by that of the symmetric lowest level could be a good approximation, if the symmetric level has nonzero energy. It is easy to see from the Dirac equations (47), (48) that the symmetric lowest level has a nonzero energy, when $\xi \neq 0$ in the case $G = SU(2)$ and $\det \hat{\xi} = \xi^2 - \vec{\xi}^2 \neq 0$ in the case $G = SU(4)$. Thus for solutions with nonstatic YM fields this approximation is quite adequate almost everywhere. Moreover, this reasoning gives an explanation to the problem of the "ghost solutions" in ED systems. It makes clear that, in the absence of gauge fields, the symmetric level of the Dirac field always has zero energy. Therefore, the corresponding energy-momentum tensor vanishes and cannot serve as an approximation for describing the contribution of the fermions on excited levels.

Now that we have explained the physical meaning of our classical spinor field, we will discuss our solutions in more detail. We start with the simpler case of the gauge group $SU(2)$. In this case the Yang-Mills equations admit three static solutions: two minima and one local maximum of the potential

$$V(\xi) = \frac{1}{8}(\xi^2 - 1)^2 + 2g^2\xi(C_u - C_v)\,. \tag{86}$$

Of course, these are solutions of the whole system only if the constant E is equal to the value of $V(\xi)$ in these extrema. In the absence of the spinor field, these extrema are two vacua $\xi = -1$, $\xi = +1$ of the YM field with Chern-Simons numbers $q = 0$ and $q = 1$ and a sphaleron-like solution $\xi = 0$ with Chern-Simons number $q = 1/2$ lying on top of the potential barrier between the vacua, where the Chern-Simons number is defined for any t as

$$q(t) = \frac{1}{8\pi^2}\int_{S^3} \mathrm{tr}_\delta(\mathrm{F}\wedge\mathrm{F}),$$

the trace being taken in the representation δ of the gauge group in the spinor space. In the case of symmetric gauge fields this number can be calculated explicitly to be $q(t) = (1/4)(2 + 3\xi(t) - (\xi(t))^3)$. Chiral spinor field configuration shifts the position of the extrema and the Chern-Simons index of the corresponding gauge field configurations, which is quite natural in the presence of matter fields [14, 15, 16, 17, 19].

By fine tuning the cosmological constant Λ and the energy E, we can get a static sphaleron-like solution of the whole system. This solution corresponds to the local maxima of $V(\xi)$ (58) and $W(a)$ (55). Obviously, this solution has two unstable modes – one in the gravitational and one in the gauge field sector. This is another indication that the static solution is a cosmological analogue of the first Bartnik-McKinnon solution [20].

Next we observe that equation (44) is the $(0,0)$ component of the Einstein equations, and therefore we can interpret the constant E as the total energy of the YMD system. Then equation (57) describes, in particular, the exchange of energy between Yang-Mills and spinor fields. This interpretation is in agreement with the solutions (63) and (64) for the spinor field in the case of the gauge group $SU(2)$: the momentary frequency resp. energy of the spinor field is given by the integrand in the exponent of the solutions and this is up to a factor ξ.

At this point we encounter the problem of positive definiteness of the energy, which is well known in classical Dirac theory. We see that the left handed and the right handed fermions make contributions of opposite sign to the energy of the system (57). Therefore, a physically meaningful solution in our case can only be a solution with either left handed or right handed fermions and the Yang-Mills field ξ taking values in the interval, where the energy of the corresponding fermion field is positive. Thus, the solution (62) with $C_u = 0$ is physically meaningful, because $\xi < 0$ for this solution and the energy of the left handed fermion field v is positive definite, which is also seen from equation (57).

If the total energy of the YMD system is larger than the local maximum of the potential $V(\xi)$, the motion in the variable ξ, stemming from the YM field, will go over an interval, including both minima of the potential $V(\xi)$. When ξ crosses zero, the energy of the spinor field also changes its sign. This is in accordance with the observations in [6, 7], where it was shown that the spinor field has zero modes in the sphaleron background and that moving between neighbouring vacua of the gauge field results in a shift of the energy level of the spinor field. We also obtain from our solution that the effect is opposite for the spinor fields with opposite chirality. But the validity of our approximation fails at this point, and we cannot consistently describe the process of creation and annihilation of fermions. This is also easily seen from the fact that the classical axial current of the Dirac field, given in the case under consideration by

$$j = \bar{\psi}\gamma_\alpha\gamma^5\psi\theta^\alpha = -\bar{\psi}\gamma_\alpha\psi\theta^\alpha == -\frac{C_v}{a^3}\theta^0\,, \tag{87}$$

is classically conserved, i.e. $d(*j) = 0$, whereas the anomaly equation [14, 15, 18, 19]

$$d(*j) = -(1/8\pi^2)\mathrm{tr}_\delta(\mathrm{F}\wedge\mathrm{F}) \tag{88}$$

implies that the evolution of the YM field between topologically distinct vacua must result in a change of the fermion number

$$N_L(t) = -\int_{\{t\}\times S^3} *j. \tag{89}$$

Thus, in our approximation only the motion in the vicinity of a certain vacuum is allowed.

Finally, we comment on the case of gauge group $SU(4)$. Under assumption (67), we can completely solve the field equations. We get two spinor fields with opposite chirality and equal density ($C_u = C_v$), see equations (82) and (83), resp. (84) and (85).

Equation (81) means that the Yang-Mills potential takes values only in an $su(2) \oplus su(2)$ subalgebra of $su(4)$ and, therefore, splits into two parts. Each part is coupled to one spinor field of a definite chirality. Thus, in some sense, we simply have a doubling of the solution for $SU(2)$.

Similar to (57), equations (78) and (79) describe the exchange of energy between spinor and gauge field. In the absence of the spinor field, the YM field has the following extrema:

a local maximum

$$\xi = 0 \qquad \zeta = 0 \qquad q = 1;$$

four minima

$$\begin{array}{lll} \xi = 1 & \zeta = 0 & q = 2 \\ \xi = 0 & \zeta = \pm 1 & q = 1 \\ \xi = -1 & \zeta = 0 & q = 0; \end{array}$$

and four saddle points

$$\begin{array}{lll} \xi = \frac{1}{2} & \zeta = \pm\frac{1}{2} & q = \frac{3}{2} \\ \xi = -\frac{1}{2} & \zeta = \pm\frac{1}{2} & q = \frac{1}{2}, \end{array}$$

the Chern-Simons number now being given by $q(t) = (1/2)(2 + 3\xi(t) - (\xi(t))^3 - 3\xi(t)(\zeta(t))^2)$.

The inclusion of the spinor field with sufficiently small $\mid S \mid$ lifts the degeneracy so that three of the four absolute minima become local minima. Again by fine tuning the constants E_1 and E_2 and the cosmological constant Λ we can obtain the corresponding static solutions, some of them being stable and some unstable.

If the constants E_1 and E_2 are such that the variables x and y describe a motion in the vicinity of the local minimum, where the energy of both left handed and right handed fermions is positive, the corresponding solutions are physically meaningful. Such solutions exist for any sign of the constant S (74). Thus, in contrast to the $SU(2)$-case, we have - due to constraints (68) and (69) - only solutions with equal density of left and right handed fermions, i.e. we have no chiral solutions.

If the energies E_1 and E_2 are large enough, in the case of gauge group $SU(4)$ we also have the phenomenon of energy level crossing in the evolution of the spinor field. But similar to the case of the gauge group $SU(2)$, these solution fail, when zero modes appear. And therefore they are unable to describe transitions between the topologically distinct vacua, which the system has in this case.

In the end we would like to emphasize that the obtained solutions are interesting in their own right as the first example of exact self-consistent solutions in Einstein-Yang-Mills-Dirac system, and it is quite possible that there can exist other interpretations of these solutions, which differ from the one proposed here.

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