Nonperturbative contributions to the QCD running coupling constant

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Abstract

We consider briefly the "analytization procedure" which allows one to remove nonphysical singularities of the QCD running coupling constant $\bar{\alpha}_s(q^2)$ in the infrared region. This procedure is applied to standard as well as to iterative solutions of the two-loop renormalization group equation. Non-leading at large momentum nonperturbative contributions in $\bar{\alpha}_s(q^2)$ are obtained in an explicit form. The coefficients of nonperturbative contributions expansions in inverse powers of the squared Euclidean momentum are calculated. The conclusion is made on an incomplete fixing of the perturbation theory summation ambiguity within "(forced) analytization procedure". With one-loop analytic coupling taken as an input a minimal model for $\bar{\alpha}_s(q^2)$ is considered so that the perturbative time-like discontinuity is preserved and nonperturbative terms not only remove the Landau singularity but also provide the ultraviolet convergence of the gluon condensate. Within this model, on the one hand, the gluon zero modes are enhanced (the dual superconductor property of the QCD vacuum) and, on the other hand, dynamical gluon mass generation is realized, with m_g estimated as $0.6GeV$. The consequences for m_g with two-loop analytic coupling taken as an input are considered.

The nonperturbative contributions to observables as well as direct to the QCD running coupling constant $\bar{\alpha}_s$ have been widely discussed in recent years. The evidence provided by very different approaches suggests the existence of power type nonperturbative corrections which reflect the complicated structure of the QCD vacuum.

The nonperturbative contributions arise quite naturally when one uses the "analytization procedure". The main purpose of this procedure is to remove nonphysical singularities from approximate (perturbative) expressions for the Green functions of QFT. The idea of the procedure goes back to Refs. [1, 2] devoted to the ghost pole problem in QED. The foundation of the procedure is the principle of summation of imaginary parts of the perturbation theory terms. Then, the Källen – Lehmann spectral representation results in the expressions without nonphysical singularities. In recent papers [3, 4] it is suggested to solve the ghost pole problem in QCD demanding the $\bar{\alpha}_s(q^2)$ be analytical in q^2 (to compare with dispersive approach [5]). As a result, instead of the one-loop expression $\bar{\alpha}_s^{(1)}(q^2) = (4\pi/b_0)/\ln(q^2/\Lambda^2)$ taking into account the leading logarithms and having the ghost pole at $q^2 = \Lambda^2$ (q^2 is the Euclidean momentum squared), one obtains the expression

$$\bar{\alpha}_{an}^{(1)}(q^2) = \frac{4\pi}{b_0}\left[\frac{1}{\ln(q^2/\Lambda^2)} + \frac{\Lambda^2}{\Lambda^2 - q^2}\right]. \tag{1}$$

Eq. (1) is an analytic function in the complex q^2-plane with a cut along the negative real semiaxis. The pole of the perturbative running coupling at $q^2 = \Lambda^2$ is canceled by the nonperturbative contribution ($\Lambda^2 \mid_{g^2 \to 0} \simeq \mu^2 \exp\{-(4\pi)^2/(b_0 g^2)\}$) and the value $\bar{\alpha}_{an}^{(1)}(0) = 4\pi/b_0$ appeared finite and independent of Λ. The nonperturbative contribution in Eq. (1) can be presented as convergent at $q^2 > \Lambda^2$ constant signs series in the inverse powers of the momentum squared.

The most important feature of the "analytization procedure" discovered is the stability property of the value of the "analytically improved" running coupling constant at zero with respect to high corrections, $\bar{\alpha}_{an}^{(1)}(0) = \bar{\alpha}_{an}^{(2)}(0) = \bar{\alpha}_{an}^{(3)}(0)$. This property provides a definite high corrections stability of $\bar{\alpha}_{an}(q^2)$ in the whole infrared region.

In this paper we study what happens with power corrections of nonperturbative ultraviolet "tail" of the analytic running coupling when going from the one-loop level to the two-loop level. Then we consider a model for $\bar{\alpha}_s(q^2)$ with additional nonperturbative terms taken into account.

Let us consider the two-loop β-function

$$\beta(g^2) = \beta_0 g^4 + \beta_1 g^6, \tag{2}$$

where the coefficients

$$\beta_0 = -\frac{1}{(4\pi)^2} b_0, \;\; b_0 = 11 - \frac{2}{3} n_f,$$

$$\beta_1 = -\frac{1}{(4\pi)^4} b_1, \;\; b_1 = 102 - \frac{38}{3} n_f \tag{3}$$

do not depend on the renormalization scheme choice. We can present the Gell-Mann – Low equation

$$u \frac{\partial \bar{g}^2(u,g)}{\partial u} = \beta(\bar{g}^2), \tag{4}$$

$u = q^2/\mu^2$ in a "rationalized" form by introducing the function $a(x)$ of $x = q^2/\Lambda^2$ of the form

$$a(x) = \frac{b_0}{(4\pi)^2} \bar{g}^2(u,g) = \frac{b_0}{4\pi} \bar{\alpha}_s(q^2).$$

The two-loop differential equation for the running coupling constant can be written as follows:

$$x \frac{da(x)}{dx} = -a^2(x) - b a^3(x), \tag{5}$$

where

$$b = -\frac{\beta_1}{\beta_0^2} = \frac{102 - \frac{38}{3} n_f}{(11 - \frac{2}{3} n_f)^2}. \tag{6}$$

For the case of three active quark flavors $n_f = 3$, $b_0 = 9$, and $b = 64/81 \simeq 0.7901$. Integrating Eq. (5) one obtains

$$\frac{1}{a(x)} - b \ln\left(1 + \frac{1}{b a(x)}\right) = \ln x + \ln c, \tag{7}$$

where c being the integration constant. The transcendental Eq. (7) defines implicitly the function $a(x)$. The real solution of Eq. (7) has two branches, the former corresponds to a positive decreasing at $x \to \infty$ function $a(x)$, whereas for the latter $a(x)$ is negative and at $x \to \infty$ it goes to $-1/b$. The positive decreasing at $x \to \infty$ solution corresponds to the asymptotic freedom property, and this solution can be fixed further by the choice of the integration constant c, which means, in practice, the redefinition of Λ. The exact solution of Eq. (7) can be written [6] using the Lambert function $W(y)$ defined by the equation $y = W(y) \exp\{W(y)\}$.

We shall consider the following two forms of approximate solutions of Eq. (7). The first one is a standard two-loop coupling

$$a^{(2)}(x) = \frac{1}{l} - \frac{b}{l^2} \ln l, \tag{8}$$

where $l = \ln x$, and the second one is the iterative solution [7] of the form

$$a^{it}(x) = \frac{1}{l + b \ln\left(1 + \frac{l}{b}\right)}. \tag{9}$$

This solution, when expanding in inverse powers of logarithms apart from the terms Eq. (8), gives rise to the term $b \ln b / l^2$ and additional terms of the form $l^{-3} \ln^2 l + ...$ which should be taken into account at the three-loop level. The exact solution of Eq. (7) also does not claim to be the description of the three-loop terms. At large x functions (8), (9) behave similarly but for small x their behavior is different and at $x = 1$ they have singularities of a different analytical structure. Namely, at $x \simeq 1$

$$a^{(2)}(x) \simeq -\frac{b}{(x-1)^2} \ln(x-1), \;\; a^{it}(x) \simeq \frac{1}{2(x-1)}. \tag{10}$$

These singularities are stronger than the integrable singularity of the exact solution of Eq. (7) of the form $1/\sqrt{x - 1/c}$. But this is not an obstacle to the application of the analytical approach.

According to the definition the analytical running coupling is obtained from the initial one by the integral representation

$$a_{an}(x) = \frac{1}{\pi}\int\limits_0^\infty \frac{d\sigma}{x+\sigma}\rho(\sigma), \tag{11}$$

where $\rho(\sigma) = \mathrm{Im}\, a_{an}(-\sigma - i0) = \mathrm{Im}\, a(-\sigma - i0)$. It is seen that dispersively-modified coupling of the form (11) has analytical structure which is consistent with causality. Of interest are nonperturbative contributions which decrease at $x \to \infty$ much faster than the perturbative ones. Therefore, their extraction by a direct comparison with the initial expressions is rather complicated at large x. The explicit analytic formulae for the nonperturbative contributions as a difference between output and input expressions of the analytization procedure are obtained in [8]. The analytical coupling can be derived with the result

$$a_{an}^{(2)}(x) = \frac{1}{\ln x} - b\frac{\ln(\ln x)}{(\ln x)^2} - \frac{1}{x-1} - b\left\{\int\limits_0^1 \frac{dt}{t^2}\left[\frac{1}{xe^t-1} - \frac{1}{x-1} + \frac{tx}{(x-1)^2}\right] + \right.$$

$$\left. + \int\limits_1^\infty \frac{dt}{t^2}\frac{1}{xe^t-1} - \frac{1}{x-1}\right\}. \tag{12}$$

Integrating by parts and making the change of variable $\sigma = \exp(-t)$, we can represent the nonperturbative part of (12) in the form of one integral with finite limits

$$a_{an}^{(2)\,npt}(x) = \frac{1}{1-x} + b\left\{\frac{x}{(1-x)^2} + x\int\limits_0^1 d\sigma\, \ln(-\ln\sigma)\frac{x+\sigma}{(x-\sigma)^3}\right\}. \tag{13}$$

Let us now turn to the iterative coupling. We obtain

$$a_{an}^{it}(x) = \frac{1}{\ln x + b\ln\left(1+\frac{1}{b}\ln x\right)} + \frac{1}{2(1-x)} +$$

$$+ \int\limits_0^\infty d\xi\, \frac{1}{1-xe^{b(1+\xi)}}\frac{1}{(1+\xi-\ln\xi)^2+\pi^2}. \tag{14}$$

Performing the change of variable $\sigma = \exp(-\xi)$, we come to the integral with finite limits. For additional "nonperturbative" output of the analytical approach we obtain the following representation

$$\Delta a_{an}^{it}(x) = \frac{1}{2(1-x)} + \int\limits_0^1 \frac{d\sigma}{\sigma}\frac{1}{1-x(\sigma/e)^{-b}}\frac{1}{[1-\ln(-\sigma\ln\sigma)]^2+\pi^2}. \tag{15}$$

Consider the large x behavior of the nonperturbative contributions. As seen from Eq. (12) this behavior is regular. Expanding the integrands in powers of $1/x$ and integrating, we find

$$a_{an}^{(2)\,npt}(x) = \sum_{n=1}^\infty \frac{c_n}{x^n}, \tag{16}$$

where

$$c_n = -1 + bn\left[n\int\limits_0^\infty dt\, \ln te^{-nt} + 1\right]. \tag{17}$$

The t integration in (17) yields the following simple formula for the coefficients of the ultraviolet expansion of the nonperturbative contributions

$$c_n = -1 + bn\,(1-\gamma-\ln n)\,, \tag{18}$$

where γ is the Euler constant, $\gamma \simeq 0.5772$. The leading term is

$$c_1 = -1 + b(1-\gamma), \tag{19}$$

and passing from the one-loop level to the two-loop one results in some compensation of the leading at large x term of the form $1/x$. For the next terms there is no compensation, instead they increase with n increasing. Estimating at $n_f = 3$ the first three terms of the series (16), we obtain

$$a_{an}^{(2)\,npt}(x) \simeq -\frac{0.666}{x} - \frac{1.427}{x^2} - \frac{2.602}{x^3} - \dots . \tag{20}$$

Let us consider the large x behavior of the "nonperturbative" contribution in (14). Expanding the integrand in powers of $1/x$, we also obtain the infinite terms series

$$\Delta a_{an}^{it}(x) = \sum_{n=1}^{\infty} \frac{c_n^{it}}{x^n}, \tag{21}$$

where

$$c_n^{it} = -\frac{1}{2} - \int\limits_0^{\infty} d\xi\, e^{-bn(1+\xi)} \frac{1}{\left[1+\xi-\ln\xi\right]^2 + \pi^2}. \tag{22}$$

Expression (22) for the expansion coefficients is not so simple as (18) but it is explicit and can be calculated up to an arbitrary accuracy. We see from Eq. (22) that all c_n^{it} are also negative tending fast to $-1/2$ when n increases. At $n_f = 3$ for the first three terms of the series (21), one can obtain

$$\Delta a_{an}^{it}(x) \simeq -\frac{0.535}{x} - \frac{0.508}{x^2} - \frac{0.502}{x^3} - \dots . \tag{23}$$

Thus, for the iterative solution all terms of expansion (21) for the two-loop case are approximately two times less then the corresponding terms for the one-loop case.

For both cases considered the nonperturbative contributions can be presented in the form of series in inverse powers of the momentum squared, all the terms are negative. Although the convergence radii of the series (16), (21) are the same and equal to unity, the dependence of the expansion coefficients on term numbers is different. This is defined by the expressions taken as an input for the analytization procedure. Choosing as an input the standard two-loop expression (8), we obtain the "standard nonperturbative corrections" of the form (20). For sufficiently large $x = q^2/\Lambda^2$ the nonperturbative contributions are defined mainly by the leading terms in (20), (23). The partial compensation of the leading terms for both cases considered when we pass from the one-loop level to the two-loop level is found.

We see that the expressions for the nonperturbative contributions have the behavior $1/q^2$ as $q^2 \to \infty$. To answer the question whether this behavior is admissible, let us consider an important physical quantity, namely, the gluon condensate $K = < vac \mid \alpha_s/\pi : F_{\mu\nu}^a\, F_{\mu\nu}^a :\mid vac >$. Normal ordering of the operator product is defined with respect to perturbative vacuum $|pert>$, as the averaging in the expression for the condensate is carried out with true physical vacuum state $|vac>$. Up to the quadratic approximation in the gluon fields, the gluon condensate K is

$$K = \lim_{x\to y} < vac|\frac{\alpha_s}{\pi} : F_{\mu\nu}^a(x) F_{\mu\nu}^a(y) : |vac> \approx \tag{24}$$

$$= \frac{3}{\pi^5} \int dk\, [\bar{\alpha}_s(k^2) - \bar{\alpha}_s^{pt}(k^2)] = \frac{3}{\pi^3} \int\limits_0^{\infty} dy\, y \bar{\alpha}_s^{npt}(y). \tag{25}$$

The corresponding chain of equations explaining this result can be found in [9]. The one-loop "analytized" behavior of Eqs. (16), (21) leads to a quadratic divergence in Eq. (25) at infinity and this is true for the two- and three-loop expressions [3] of the analytization approach. According to the results of Refs. [10, 11, 12], let us add in Eq. (1) the isolated infrared singular term of the form $1/q^2$. This term is harmless at zero and it can improve the behavior of the integrand at infinity and make the integral logarithmically divergent. To make the integral (25) convergent at infinity, it is sufficient to add one more isolated singular term of

a pole type with parameters chosen appropriately. In this sense the model we come to is minimal. The expression we obtain for the running coupling constant is the following [13, 14]:

$$\bar{\alpha}_s(q^2) \;=\; \frac{4\pi}{b_0}\left(\frac{1}{\ln(q^2/\Lambda^2)} \;+\; \frac{\Lambda^2}{\Lambda^2-q^2} \;+\; \frac{c\Lambda^2}{q^2} \;+\; \frac{(1-c)\Lambda^2}{q^2+m_g^2}\right), \tag{26}$$

with mass parameter $m_g = \Lambda/\sqrt{c-1}$. It is worth noting that taking account of nonperturbative contributions in Eq. (26) preserves a perturbative time-like cut of Eq. (1). With the given value of the QCD scale parameter Λ, the parameter c can be fixed by the string tension κ, assuming the linear confinement $V(r) \simeq \kappa r = a^2 r$ at $r \to \infty$. We define the potential $V(r)$ used as a static $q\bar{q}$ interaction, by means of the three-dimensional Fourier transform of $\bar{\alpha}_s(\vec{q}^{\,2})/\vec{q}^{\,2}$, with the contributions of only one dressed gluon exchange taken into account. This gives the relation $c\Lambda^2 = (3b_0/8\pi)a^2$. Taking $a \simeq 0.42\,GeV$, one obtains $c = \Lambda_1^2/\Lambda^2$ where $\Lambda_1^2 = 3b_0\kappa/8\pi \simeq (0.434\,GeV)^2$ ($b_0 = 9$ in the case of 3 light flavors). For m_g one obtains

$$m_g \;=\; \Lambda^2/\sqrt{\Lambda_1^2-\Lambda^2}, \tag{27}$$

and the tachion absence condition limits the parameter Λ to $\Lambda < 434MeV$.

In this case the model for the running coupling constant arises with the enhancement at zero momentum and dynamical gluon mass m_g. This mass (as well as Λ) can be fixed [9] by the condition of minimum of the nonperturbative vacuum energy and is estimated as $m_g \simeq 0.6$ GeV (and $\Lambda \simeq 375$ MeV) for the "standard" value of the gluon condensate $(0.33\ \mathrm{GeV})^4$. The next step of the model building for the full running coupling constant on the base of the two-loop analytical running coupling gives:

$$\bar{\alpha}_s(q^2) \;=\; \bar{\alpha}_{s\ an}(q^2) + \frac{4\pi}{b_0}\left(\frac{\Lambda_1^2}{\Lambda^2} \;-\; \frac{(\Lambda_1^2+c_1\Lambda^2)^2}{q^2(\Lambda_1^2+c_1\Lambda^2)-c_2\Lambda^4}\right). \tag{28}$$

For the dynamical gluon mass one obtains:

$$m_g \;=\; \Lambda^2\sqrt{\frac{-c_2}{\Lambda_1^2+c_1\Lambda^2}}. \tag{29}$$

For the "standard" two-loop input and $\Lambda \simeq 375$ MeV we obtain close value $m_g \simeq 546$ MeV, and if we start with iterative solution the dynamical gluon mass is essentially lower, $m_g \simeq 298$ MeV.

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