One Kind of Spinor Techniques for Massive Fermions

V. V. Andreev [1]

Gomel State University, Phys.Department, Sovietskya str. 102, Gomel, 246699
Belarus

Abstract

In this work a techniques of reduction the matrix element of a process with massive fermions to spinor products is considered. Our procedure is not more complicated than the CALCUL spinor techniques for massless fermions. It is relatively simple and leads to the basic spinor products with arbitrary polarization of fermions. Also we obtained "spinor" Chisholm identities for massive fermions.

The study of the high-energy processes with polarization are of fundamental importance in understanding the structure of matter. For example, The European Muon Collaboration (EMC) and SLAC experiments [1] on deep inelastic scattering of longitudinally polarized muons on longitudinally polarized target has catalyzed an extraordinary outburst of theoretical activity.

With increasing energy of colliders, processes involving many final-state particles ($2 \to 3$, $2 \to 4$, $2 \to 5$, ...) are important path the present collider's physics.

The calculation of cross sections for these processes becames difficult if one uses conventional approaches. In these approaches the Feynman amplitude has to be squared and because of this they are very inconvenient if the number of Feynman diagrams and the number of final state particles are large.

An alternative approach is to compute the Feynman amplitudes symbolically or numerically. The idea of calculating Feynman amplitudes is as old as conventional approach. For example, covariant method of calculating amplitudes have been developed more than thirty years ago [2]-[4].

Now there exist many different methods. In particular, the convenient techniques for massless external fermions was introduced by the CALCUL collaboration [5]-[7]. Although, there have been several attempts to generalize the method developed by the CALCUL collaboration for massless fermions to massive case, the resulting procedure is rather cumbersome and requires introduction of arbitrary reference momenta [6].

It is important to note, that few methods of analytical calculations of reactions with massive fermions are convenient for a realization on a computer. By such, except the above mentioned method of the group CALCUL, it is necessary to relate methods offered in ref. [8],[9]. In ref.[8] the compact formalism of evaluation of matrix elements based on insertion in spinor lines of a complete set states build up unphysical spinors is offered, that has allowed to create the high speed program for evaluations of Feynman amplitudes.

It is purpose of this paper to present a simple method for calculation of amplitudes with using of spinor products, which works for processes involving massive fermions. The approach, that we present here leads to an expression for a Feynman amplitude in terms of spinor products.

CALCUL spinor techniques for massless fermions

Let the four-vectors

$$n_0 = (1,0,0,0)\ , n_1 = (0,1,0,0),\ n_2 = (0,0,1,0),\ n_3 = (0,0,0,1)$$

are the basis orts in Minkowski space. By using this vectors we can obtain light-like vectors

$$b_0 = n_0 - n_3, b_3 = n_0 + n_3, b_\lambda = n_1 + i\,\lambda n_2, \quad \lambda = \pm 1$$

with following properties:

$$b_0 \cdot b_\lambda = 0,\ b_3 \cdot b_\lambda = 0,\ b_0 \cdot b_3 = 2,\ b_+ \cdot b_- = -2.$$

Next we define *"basic spinor"* $U_\lambda(b_0)$ up to its complex phase by specifying the corresponding projection operator and phase condition:

$$U_\lambda(b_0)\,\overline{U}_\lambda(b_0) = \omega_\lambda \not{b}_0, \tag{1}$$

[1] E-mail: andreev@gsu.unibel.by

$$\frac{\lambda}{2}\not{b}_\lambda U_{-\lambda}(b_0) = U_\lambda(b_0) \tag{2}$$

with the matrix $\omega_\lambda = 1/2\,(1+\lambda\gamma_5)$.

The CALCUL spinor techniques for calculating processes with massless external fermions involves the following operations.

1 step. From (1)-(2) we construct spinors of fermions (or antifermions) for any light-like momentum p as follows

$$U_\lambda(p) = \not{p}\, U_{-\lambda}(b_0)\,/\sqrt{2\,p.b_0}. \tag{3}$$

2 step. By using so-called 'spinor' Chisholm identity

$$\gamma_\mu\left\{\overline{U}_\lambda(p)\,\gamma_\mu U_\lambda(k)\right\} =$$

$$= 2\,U_\lambda(k)\,\overline{U}_\lambda(p) + 2\,U_{-\lambda}(p)\,\overline{U}_{-\lambda}(k) \tag{4}$$

and the equation for any four-vector p with $p^2 = 0$

$$\not{p} = \sum_\lambda U_\lambda(p)\,\overline{U}_\lambda(p)\,, \tag{5}$$

we can reduce many amplitudes to expressions involving only spinor products $\overline{U}_\lambda(p)$ and $U_{-\lambda}(k)$ with for vectors particles interacting in process. The remaining possible spinor products are vanishing due to helicity conversation. Spinor products for antifermions with $V_\lambda(p)$ are always referred to their fermion counterpart $U_{-\lambda}(p)$ through the equation $V_\lambda(p) = U_{-\lambda}(p)$.

Two nonzero products

$$s(p,k) = \overline{U}_+(p)\,U_-(k) = -s(k,p)\,,$$
$$t(p,k) = \overline{U}_-(p)\,U_+(k) = s^*(k,p) \tag{6}$$

due to equations (1)-(2) reduce to trace

$$\overline{U}_\lambda(p)\,U_{-\lambda}(k) = \lambda/\left(\sqrt{b_0\cdot p}\sqrt{b_0\cdot k}\right)\frac{1}{4}\,Tr\left(\omega_{-\lambda}\not{b}_0\not{p}\not{k}\not{b}_\lambda\right) =$$

$$\lambda/\left(2\sqrt{b_0\cdot p}\sqrt{b_0\cdot k}\right)\left[p\cdot b_0\,k\cdot b_\lambda - k\cdot b_0\,p\cdot b_\lambda - i\epsilon_{\mu\nu\rho\sigma}b_0^\mu\,b_\lambda^\nu\,p^\rho\,k^\sigma\right]. \tag{7}$$

For light-like vectors $p = (p_0, p_x, p_y, p_z)$ and $k = (k_0, k_x, k_y, k_z)$ we can obtain

$$s(p,k) = \sqrt{p_-k_+}\exp(i\varphi_p) - \sqrt{p_+k_-}\exp(i\varphi_k) \tag{8}$$

with $p_\pm = p_0 \pm p_z$, $p_x \pm ip_y = \sqrt{p_+p_-}\exp(\pm i\varphi_p)$. The spinor product (8) of two four-vectors is not complicated as compared to a simple scalar Minkowski product $p\cdot k$.

Generalizations of the CALCUL approach to the case of massive fermions are exist [6]-[7], but the results are cumbersome: the method requires the introduction of extra light-like momenta and expression of the spinor for massive fermion as sum over spinors with extra momenta. We show below, that in the framework of our approach, it is possible to write down the amplitude in terms of spinor products for any polarization vector of massive fermions.

Spinor techniques for massive fermions

Our procedure is similar to that described in the previous section. Let consider bispinors, which related with the "basic spinor" (2) as follows

$$U_\lambda(p,s_p) = \frac{\tau_u^\lambda(p,s_p)}{\sqrt{b_0.(p+m_ps_p)}}U_{-\lambda}(b_0)\,, \tag{9}$$

$$V_\lambda(p,s_p) = \frac{\tau_v^\lambda(p,s_p)}{\sqrt{b_0.(p+m_ps_p)}}U_\lambda(b_0)\,, \tag{10}$$

where the projection operators $\tau_u^{\lambda}(p,s_p)\,,\tau_u^{\lambda}(p,s_p)$ are

$$\tau_u^{\lambda}(p,s_p)=\frac{1}{2}\left(\not{p}+m_p\right)\left(1+\lambda\,\gamma_5\ \not{s}_p\right), \tag{11}$$

$$\tau_v^{\lambda}(p,s_p)=\frac{1}{2}\left(\not{p}-m_p\right)\left(1+\lambda\gamma_5\ \not{s}_p\right). \tag{12}$$

We can get that

$$\begin{gathered}\not{p}\,U_{\lambda}(p,s_p)=m_p\,U_{\lambda}(p,s_p)\,,\not{p}\,V_{\lambda}(p,s_p)=-m_p\,V_{\lambda}(p,s_p)\,,\\ \gamma_5\ \not{s}_p\,U_{\lambda}(p,s_p)=\lambda\,U_{\lambda}(p,s_p)\,,\gamma_5\ \not{s}_p\,V_{\lambda}(p,s_p)=\lambda\,V_{\lambda}(p,s_p)\end{gathered} \tag{13}$$

i.e. the bispinors $U_{\lambda}(p,s_p)$ and $V_{\lambda}(p,s_p)$ satisfies the Dirac equation and spin conditions for a massive fermion and antifermion. We also can get, that bispinors of fermions and antifermions (9)-(10) are related as follows

$$V_{\lambda}(p,s_p)=-\lambda\gamma_5 U_{-\lambda}(p,s_p)\,,\overline{V}_{\lambda}(p,s_p)=\overline{U}_{-\lambda}(p,s_p)\,\lambda\,\gamma_5. \tag{14}$$

Let to count the number of spinor products for massive fermions. It is obvious, that from the technical point of view the situation with massive fermions is more difficult as compared to the massless case. In the most general case the calculation of matrix elements requires sixteen spinor products instead of two in the massless version. And consequently, on the first sight, it seems, that the use of a reduction of a matrix element of the process to spinor products is not absolutely convenient. However, here it is possible to achieve essential simplification. At first, all spinor products are not linearly independent. Using ratios (14), that only eight are linearly independent. We define *"basic spinor products"* for massive fermions as

$$\begin{gathered}\overline{U}_{\lambda}(p,s_p)\,U_{-\lambda}(k,s_k)\,,\\ \overline{U}_{\lambda}(p,s_p)\,U_{\lambda}(k,s_k)\,,\\ \overline{V}_{\lambda}(p,s_p)\,U_{\lambda}(k,s_k)\,,\\ \overline{V}_{\lambda}(p,s_p)\,U_{-\lambda}(k,s_k)\,.\end{gathered} \tag{15}$$

Secondly, basic spinor products can be calculated with the help of second functions. Recall, that γ_5 for massless bispinor is the operator of a spin i.e.

$$\gamma_5 U_{\lambda}(b_0)=\lambda U_{\lambda}(b_0)\,. \tag{16}$$

Then the definition of bispinors (9)-(10) is possible to rewrite as

$$U_{\lambda}(p,s_p)=\frac{\chi(p,s_p,+1)}{2\sqrt{b_0.\,(p+m_p s_p)}}U_{-\lambda}(b_0)\,, \tag{17}$$

$$V_{\lambda}(p,s_p)=\frac{\chi(p,s_p,-1)}{2\sqrt{b_0.\,(p+m_p s_p)}}U_{\lambda}(b_0) \tag{18}$$

with functions

$$\chi(p,s_p,a)=\left(\not{p}+a\,m_p\right)\left(1+a\ \not{s}_p\right). \tag{19}$$

Let introduce the following functions:

$$s\chi(p,k,s_p,s_k,\lambda,a,b)=\lambda/\left(2\sqrt{b_0.\,(p+m_p\,s_p)}\sqrt{b_0.\,(k+m_k\,s_k)}\right)$$

$$\frac{1}{4}Tr\left(\omega_{-\lambda}\ \not{b}_0\,\chi^{\dagger}(p,s_p,a)\,\chi(k,s_k,b)\ \not{b}_{\lambda}\right), \tag{20}$$

$$w\chi(p,k,s_p,s_k,\lambda,a,b)=1/\left(\sqrt{b_0.\,(p+m_p\,s_p)}\sqrt{b_0.\,(k+m_k\,s_k)}\right)$$

$$\frac{1}{4}Tr\left(\omega_{-\lambda}\ \not{b}_0\,\chi^{\dagger}(p,s_p,a)\,\chi(k,s_k,b)\right). \tag{21}$$

All "basic" spinor products are reduced to functions (20) - (21):

$$\begin{array}{c}\overline{U}_\lambda\left(p,s_p\right)U_{-\lambda}\left(k,s_k\right)=s\chi\left(p,k,s_p,s_k,\lambda,a=+1,b=+1\right),\\ \overline{U}_\lambda\left(p,s_p\right)U_{\lambda}\left(k,s_k\right)=w\chi\left(p,k,s_p,s_k,\lambda,a=+1,b=+1\right),\\ \overline{V}_\lambda\left(p,s_p\right)U_{\lambda}\left(k,s_k\right)=s\chi\left(p,k,s_p,s_k,-\lambda,a=-1,b=+1\right),\\ \overline{V}_\lambda\left(p,s_p\right)U_{-\lambda}\left(k,s_k\right)=w\chi\left(p,k,s_p,s_k,-\lambda,a=-1,b=+1\right).\end{array} \tag{22}$$

Like for massless fermions, the spinor products (22) can be calculated through components of vectors p,k,s_p,s_k. It is obvious, that this analytical expression is more complicated as compared with appropriate relations (7) - (8).

Essential simplification as in function evaluation to which the "basic" spinor products are reduced it is possible to achieve at the expense of specific selection of polarization vectors of massive fermions.

The fermion polarization vector s_p can be expressed through the fermion momentum and any other vector of the problem

$$s_p=\frac{p.q\,p-m_p^2\,q}{m_p\sqrt{(p.q)^2-m_p^2\,q^2}} \tag{23}$$

Here q is an arbitrary vector. It is easy to convince, that s_p satisfies to the standard conditions. By selection $q=b_0$ we obtain the polarization vector used in ref.[6], and if $q=n_0=(1,0,0,0)$, we shall receive the helicity state. Selecting the polarization vector s_p for a fermion (antifermion) with momentum p and mass m_p as

$$s_p=p/m_p-m_p\,\frac{b_0}{p.b_0}. \tag{24}$$

the function $\chi\left(p,s_p,a\right)$, included in the definitions of bispinors (17)-(18), receives a form

$$\chi\left(p,s_p,a\right)=2\left(\not{p}+a\,m_p\right), \tag{25}$$

and the factor becomes equal $2\sqrt{2\,b_0.p}$.

It, queue results in sharp simplification of expressions for the functions (20)-(21) through components of vectors. These functions now do not depend on polarization vectors of fermions and can be calculated with the help of following formulas:

$$s\chi\left(p,k,s_p,s_k,\lambda,a,b\right)\equiv s_\lambda\left(p,k\right)=$$

$$\lambda/\left(2\sqrt{b_0.p}\sqrt{b_0.k}\right)\left[p.b_0\;k.b_\lambda-k.b_0\;p.b_\lambda-i\lambda\epsilon_{\mu\nu\rho\sigma}b_0^\mu\;b_\lambda^\nu\;p^\rho\;k^\sigma\right], \tag{26}$$

$$w\chi\left(p,k,s_p,s_k,\lambda,a,b\right)\equiv w\left(p,k,a,b\right)=$$

$$1/\left(\sqrt{b_0.p}\sqrt{b_0.k}\right)\left[a m_p\;k.b_0+b m_k\;p.b_0\right]. \tag{27}$$

The formula (26) does not differ from a similar ratio for the case of massless fermions (7) (only it is necessary to remember, that $p^2=m_p^2,k^2=m_k^2$). In addition, there is a function (27). To calculate matrix elements with massive fermions the group CALCUL of [6] used fermions with polarization vectors (24), but with introduction additional for each fermions light-like of momentums, that actually did a procedure of evaluations is much more complex on a comparison with similar for the massless fermions. In our case "basic" products practically differ from similar addends in version with massless fermions, but we not enter expansions of vectors of process on light-like vectors. In the massless limit, our expressions easy simplifies to spinor products $s\left(p,k\right)$ and $t\left(p,k\right)$ (6).

Chisholm indenties for massive fermions

The important stage in the transformation of matrix elements to the basis spinor products is the use of the Chisholm "spinor" identities of the type (4). For a proof of this ratio (see,[6]) the Chisholm identity for trace was used:

$$\gamma_\mu * Tr(\gamma_\mu\;\not{C}_1\ldots\not{C}_{2n+1})=$$

$$= 2\left(\not{C}_1 \ldots \not{C}_{2n+1} + \not{C}_{2n+1} \ldots \not{C}_1\right). \tag{28}$$

As the expression of the type $\overline{U}_\lambda(p,s_p)\,\gamma_\mu U_\lambda(k,s_k)$ is reduced to trace, with the use of (9), 10), (28) it is possible to receive appropriate "spinor" identities for massive fermions. In this case is present 4 (four) such "basic" of identities (as compared to one in the massless version):

$$\gamma_\mu\left\{\overline{U}_\lambda(p,s_p)\,\gamma_\mu U_\lambda(k,s_k)\right\} =$$

$$U_\lambda(k,s_k)\,\overline{U}_\lambda(p,s_p) + U_{-\lambda}(p,s_p)\,\overline{U}_{-\lambda}(k,s_k) +$$
$$+V_{-\lambda}(k,s_k)\,\overline{V}_{-\lambda}(p,s_p) + V_\lambda(p,s_p)\,\overline{V}_\lambda(k,s_k), \tag{29}$$

$$\gamma_\mu\left\{\overline{U}_\lambda(p,s_p)\,\gamma_\mu U_{-\lambda}(k,s_k)\right\} =$$

$$U_{-\lambda}(k,s_k)\,\overline{U}_\lambda(p,s_p) - U_{-\lambda}(p,s_p)\,\overline{U}_\lambda(k,s_k) +$$
$$+V_\lambda(p,s_p)\,\overline{V}_{-\lambda}(k,s_k) - V_\lambda(k,s_k)\,\overline{V}_{-\lambda}(p,s_p), \tag{30}$$

$$\gamma_\mu\left\{\overline{V}_\lambda(p,s_p)\,\gamma_\mu U_{-\lambda}(k,s_k)\right\} =$$

$$U_{-\lambda}(k,s_k)\,\overline{V}_\lambda(p,s_p) + V_{-\lambda}(p,s_p)\,\overline{U}_\lambda(k,s_k)$$
$$+V_\lambda(k,s_k)\,\overline{U}_{-\lambda}(p,s_p) + U_\lambda(p,s_p)\,\overline{V}_{-\lambda}(k,s_k), \tag{31}$$

$$\gamma_\mu\left\{\overline{V}_\lambda(p,s_p)\,\gamma_\mu U_\lambda(k,s_k)\right\} =$$

$$U_\lambda(k,s_k)\,\overline{V}_\lambda(p,s_p) + U_\lambda(p,s_p)\,\overline{V}_\lambda(k,s_k)$$
$$-V_{-\lambda}(k,s_k)\,\overline{U}_{-\lambda}(p,s_p) - V_{-\lambda}(p,s_p)\,\overline{U}_{-\lambda}(k,s_k). \tag{32}$$

The other possible combinations are reduced to above-stated with the use of (14). In case of massless fermions (29) and (31) coincide one another (with replacement $\lambda \to -\lambda$) and pass into (4), and in identities (30), (32) the right-handed terms are vanish, owing to a conservation helicity law. With the use of (29)-(32), the Dirac equation and the condition of completeness for bispinors we can write down an analytical expression of the matrix element in terms of the spinor products.

References

[1] J.Ashma et. al., Phys. Lett. **B206** (1988)1167.
[2] E.Bellomo, Nuovo Cimento **V.21** (1961) 730.
[3] A.A.Bogush, F.I.Fedorov, Vesti AN BSSR, ser.fiz.-m.n. **N2** (1962) 26.
[4] E.Bjorken and M.Chen , Phys.Rev.**V.154** (1966) 1335.
[5] F.A.Berendz, P.H.Davervedt, R.Kleiss, Nucl. Phys. **B253** (1985) 141.
[6] R.Kleiss, W.J.Stirling, Nucl. Phys. **B262** (1985) 235.
[7] R.Kleiss, Z.Phys.C -Particles and Fields **33** (1987) 433.
[8] A.Ballestrero and E.Maina, Phys.Lett.**B350** (1995) 225; Preprint DFTT 76/93,1994.
[9] K.Hagiwara, D.Zeppenfeld, Nucl. Phys. **B274** (1982)1