Exact Results on Quantum Field Theories Interpolating Between Pairs of Conformal Field Theories

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Abstract

I review recent results on conformal field theories in four dimensions and quantum field theories interpolating between conformal fixed points, supersymmetric and non-supersymmetric. The talk is structured in three parts: *i)* central charges, *ii)* anomalous dimensions and *iii)* quantum irreversibility.

The family of theories, in general UV-free, that interpolate between two conformal fixed points, in such a way that the IR limit is reachable by (resummed) perturbation theory, is called *conformal window*. The conformal window, which can be viewed as the *convergence radius* of the perturbative series, does not contain QCD, where other non-perturbative effects have to be taken into account. Yet it is the region separating the perturbative regime from QCD. Understanding the conformal window better can be a source of insight into the low-energy limit of QCD itself.

In two dimensions conformal field theories have an infinite symmetry [1] and are sometimes exactly solvable. In higher dimensions, there are simplifications in the presence of supersymmetry and exact results are available. Very general theorems, implications of unitarity, give exact results even in the absence of supersymmetry.

Here I summarize the research that I undertook on these issues over the past three years. The paper is divided in three sections: *i)* central charges, based on ref. [2]; *ii)* anomalous dimensions, on refs. [3, 4, 5]; and *iii)* quantum irreversibility, on refs. [6, 7, 8].

1. Central charges

I consider, as a concrete example, N=1 supersymmetric QCD with group $G = SU(N_c)$ and N_f quarks in the fundamental representation. I compute the infrared values of the gravitational central charges called c and a in the conformal window $3N_c/2 < N_f < 3N_c$.

The theory contains gauge superfields V^a, $a = 1, \ldots, N_c^2 - 1$, and chiral quark and antiquark superfields, $Q^{\alpha i}$ and $\widetilde{Q}_{\alpha i}$, $\alpha = 1, \ldots, N_c$, $i = 1, \ldots, N_f$, whose physical components are the gauge potentials A^a_μ and Majorana gauginos λ^a, and the complex scalars $\phi^{\alpha i}$ and $\tilde{\phi}_{\alpha i}$ and Majorana spinors $\psi^{\alpha i}$ and $\tilde{\psi}_{\alpha i}$, respectively. This theory has the usual gauge interactions and no superpotential.

The Konishi and R currents, whose fermion contributions are

$$K_\mu = \frac{1}{2}\bar{\psi}\gamma_\mu\gamma_5\psi + \frac{1}{2}\bar{\tilde{\psi}}\gamma_\mu\gamma_5\tilde{\psi}, \qquad R_\mu = \frac{1}{2}\bar{\lambda}^a\gamma_\mu\gamma_5\lambda^a - \frac{1}{6}(\bar{\psi}\gamma_\mu\gamma_5\psi + \bar{\tilde{\psi}}\gamma_\mu\gamma_5\tilde{\psi}),$$

are classically conserved, but anomalous at the quantum level. We distinguish *internal* and *external* anomalies, the latter associated with external background sources.

The internal anomalies of K_μ and R_μ are expressed by the operator equations

$$\partial_\mu R^\mu = \frac{1}{48\pi^2}[3N_c - N_f(1-\gamma)]F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}, \qquad \partial_\mu K^\mu = \frac{N_f}{16\pi^2}F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}.$$

There is an anomaly-free, RG-invariant combination of K_μ and R_μ [9]:

$$S_\mu = R_\mu + \frac{1}{3}\left(1 - \frac{3N_c}{N_f} - \gamma\right)K_\mu.$$

The coefficient of K_μ is the numerator of the exact NSVZ [10] β-function

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{3N_c - N_f(1-\gamma(g))}{1 - g^2N_c/8\pi^2} \tag{1}$$

and $\gamma/2$ is the anomalous dimension of the superfield Q (or $\widetilde{Q}$).

The first example of exact IR result is the anomalous dimension

$$\gamma_{\rm IR} = 1 - \frac{3N_c}{N_f} \tag{2}$$

of the quark fields, obtained by setting β equal to zero. We now compute other interesting quantities in the IR limit.

R_μ is the lowest component of the supercurrent superfield $J_{\alpha\dot{\alpha}}$ that also contains the stress tensor and supersymmetry currents. To study the gravitational central charges we introduce the background metric $g_{\mu\nu}$ and source V_μ for the R-current. In these background fields the trace and R-anomalies are related by supersymmetry and read, in a critical theory,

$$\Theta = \frac{c}{16\pi^2}(W_{\mu\nu\rho\sigma})^2 - \frac{a}{16\pi^2}(\widetilde{R}_{\mu\nu\rho\sigma})^2 + \frac{c}{6\pi^2}V_{\mu\nu}^2, \quad \partial_\mu R^\mu = \frac{c-a}{24\pi^2}R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma} + \frac{5a-3c}{9\pi^2}V_{\mu\nu}\widetilde{V}^{\mu\nu}.$$

We include a factor of $\sqrt{g}$ in the definition of R^μ. Here $W_{\mu\nu\rho\sigma}$ is the Weyl tensor and $\widetilde{R}_{\mu\nu\rho\sigma}$ is the dual of the curvature tensor, the second term of Θ being the Euler density; $V_{\mu\nu}$ is the field strength of V_μ. The coefficient a of the Euler density is an independent constant, while the coefficients of the $(W_{\mu\nu\rho\sigma})^2$ and $(V_{\mu\nu})^2$ terms are related. This can be proved by observing that the two-point function of $J^{\alpha\dot{\alpha}}$ has a unique structure in superspace [11],

$$\langle J_{\alpha\dot{\alpha}}(z)J_{\beta\dot{\beta}}(0)\rangle \propto c\,\frac{s_{\alpha\dot{\beta}}\bar{s}_{\beta\dot{\alpha}}}{(s^2\bar{s}^2)^2}, \tag{3}$$

and calculating the partial derivative $\mu\,\partial/\partial\mu$ of the correlators

$$\langle T_{\mu\nu}(x)\,T_{\rho\sigma}(0)\rangle = -\frac{c}{48\pi^4}\prod\nolimits^{(2)}_{\mu\nu,\rho\sigma}\left(\frac{1}{|x|^4}\right), \qquad \langle R_\mu(x)\,R_\nu(0)\rangle = \frac{c}{3\pi^4}\,\pi_{\mu\nu}\left(\frac{1}{|x|^4}\right),$$

using

$$\int\Theta = \mu\frac{\partial}{\partial\mu}, \qquad \mu\frac{\partial}{\partial\mu}\left(\frac{1}{|x|^4}\right) = 2\pi^2\delta(x). \tag{4}$$

Here $\pi_{\mu\nu} = \partial_\mu\partial_\nu - \Box\delta_{\mu\nu}$ and $\prod^{(2)}_{\mu\nu,\rho\sigma} = 2\pi_{\mu\nu}\pi_{\rho\sigma} - 3(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho})$. The second relation of (4) can be obtained by means of a regularization technique.

In a free supersymmetric gauge theory with N_v gauge and N_χ chiral multiplets, the values of c and a are

$$c_{UV} = \frac{1}{24}\left(3N_v + N_\chi\right), \quad a_{UV} = \frac{1}{48}\left(9N_v + N_\chi\right).$$

Off-criticality, there are additional terms in Θ and $\partial_\mu R^\mu$, proportional to $\beta(g)$, including the internal contribution $\beta/4\ F_{\mu\nu}^2$, and the central charges depend on the coupling, i.e. $c = c(g)$ and $a = a(g)$.

Since S_μ is quantum-conserved in the absence of sources, its external anomalies are μ-independent [12]:

$$\partial_\mu S^\mu = \frac{s_1}{24\pi^2}R_{\mu\nu\rho\sigma}\widetilde{R}^{\mu\nu\rho\sigma} + \frac{s_2}{9\pi^2}V_{\mu\nu}\widetilde{V}^{\mu\nu}.$$

A one-loop computation gives

$$s_1 = \frac{1}{16}(N_c^2+1), \qquad s_2 = \frac{9}{16}\left(N_c^2 - 1 - 2N_cN_f\left(\frac{N_c}{N_f}\right)^3\right).$$

Now, we observe that $R_\mu = S_\mu$ in the IR limit, whence

$$s_1 = c_{\rm IR} - a_{\rm IR}, \qquad s_2 = 5a_{\rm IR} - 3c_{\rm IR},$$

so that we finally get

$$c_{\rm IR}=\frac{1}{16}\left(7\,N_c^2-2-9\frac{N_c^4}{N_f^2}\right),\qquad a_{\rm IR}=\frac{3}{16}\left(2\,N_c^2-1-3\frac{N_c^4}{N_f^2}\right). \tag{5}$$

Observe that $c_{\rm IR}$ and $a_{\rm IR}$ are non-negative throughout the conformal window, in agreement with their nature of central charges. In particular, the inequality $c_{\rm IR}\geq 0$ follows from reflection positivity of the stress-tensor two-point function.

The total flows of the central charges are

$$c_{\rm UV}-c_{\rm IR}=-\frac{N_cN_f}{48}\gamma_{IR}\left(3\frac{N_c}{N_f}+9\frac{N_c^2}{N_f^2}-4\right),\quad a_{\rm UV}-a_{\rm IR}=\frac{N_cN_f}{48}\gamma_{IR}^2\left(2+3\frac{N_c}{N_f}\right)\geq 0. \tag{6}$$

The difference $a_{\rm UV}-a_{\rm IR}$ is everywhere positive in the conformal window, as conjectured by Cardy [13]. This phenomenon is called *quantum irreversibility*. Instead, the difference $c_{\rm UV}-c_{\rm IR}$ is positive in part of the conformal window and negative in the rest.

A corollary of the above derivation is that both c and a are constant on families of conformal field theories, i.e. they are marginally uncorrected.

The procedure that I have illustrated can be applied any time there is a unique R-current and a conformal window. The effects of mass perturbations and symmetry breaking can be straightforwardly included. The analysis of a wide class of models, done in [14], confirms the conclusions just derived, in particular the inequalities $a_{\rm UV}\geq a_{\rm IR}\geq 0$, on which I have more to say in section 3.

2. Anomalous dimensions

The second class of quantities that characterize a conformal field theory are the anomalous dimensions. Given that the operator-product expansion of the stress-tensor does not close, of primary interest is the spectrum of anomalous dimensions of the (infinitely many) higher-spin currents generated by the singular terms of the TT OPE. In general the classification is not simple, but with the help of supersymmetry we can reach this goal in various models. Using theorems discovered in the context of the deep inelastic scattering, in particular the Ferrara–Gatto–Grillo theorem [15] and the Nachtmann theorem [16], several conclusions about strongly coupled conformal field theories can be derived. These conclusions hold also for non-supersymmetric theories.

The algorithm to work out the currents of the quantum conformal algebra starts from the stress tensor T and the spin-0 component Σ_0 of the Konishi multiplet, which is the first operator generated by the TT OPE [11], and proceeds via a combination of two steps:

i) supersymmetry, which moves "vertically" in the algebra, i.e. changes the dimension of the operators and therefore their singularity in the OPE;

ii) orthogonalization of two-point functions, which moves "horizontally", i.e. at the same singularity level in the OPE.

I illustrate this very briefly in the case of the N=2 vector multiplet. The current multiplets have length 2 in spin units, in particular the multiplet of the stress tensor. There is one multiplet for each spin, even or odd.

The vector, spinor and scalar contributions to the currents of the N=2 vector multiplet (A_μ,λ_i,M,N), $i=1,2$, are schematically given in the free-field limit by

$$\begin{aligned}\mathcal{J}^V &= F^+_{\mu\alpha}\overleftrightarrow{\Omega}_{\rm even}F^-_{\alpha\nu},\quad \mathcal{J}^F=\frac{1}{2}\bar\lambda_i\gamma_\mu\overleftrightarrow{\Omega}_{\rm odd}\lambda_i,\quad \mathcal{J}^S=M\overleftrightarrow{\Omega}_{\rm even}M+N\overleftrightarrow{\Omega}_{\rm even}N,\\ \mathcal{A}^V &= F^+_{\mu\alpha}\overleftrightarrow{\Omega}_{\rm odd}F^-_{\alpha\nu},\quad \mathcal{A}^F=\frac{1}{2}\bar\lambda_i\gamma_5\gamma_\mu\overleftrightarrow{\Omega}_{\rm even}\lambda_i,\quad \mathcal{A}^S=-2iM\overleftrightarrow{\Omega}_{\rm odd}N,\end{aligned}$$

plus improvement terms [17], where $\overleftrightarrow{\Omega}_{\rm even/odd}$ denotes an even/odd string of derivative operators $\overleftrightarrow{\partial}$, $\mathcal{J}$,$\mathcal{A}$ denote the even and odd (axial) currents, and V,F,S mean vector, fermion, scalar. A simple set of rules determines the operation (i). The result is

$$\begin{aligned}\mathcal{J}^S &\rightarrow -2\mathcal{A}^F+2\mathcal{A}^S,\quad \mathcal{J}^F\rightarrow -8\mathcal{A}^V+\mathcal{A}^S,\quad \mathcal{J}^V\rightarrow -2\mathcal{A}^V+\frac{1}{4}\mathcal{A}^F,\\ \mathcal{A}^F &\rightarrow -8\mathcal{J}^V+\mathcal{J}^S,\quad \mathcal{A}^S\rightarrow -2\mathcal{J}^F+2\mathcal{J}^S,\quad \mathcal{A}^V\rightarrow -2\mathcal{J}^V+\frac{1}{4}\,\mathcal{J}^F.\end{aligned}$$

This operation raises the spin by one unit and it is independent of the spin on the basis ($\mathcal{J}^{S,F,V}$, $\mathcal{A}^{S,F,V}$), which is, however, not diagonal in the sense of point (ii). The diagonalization produces the correct higher-spin currents, which are rational combinations of ($\mathcal{J}^{S,F,V}$, $\mathcal{A}^{S,F,V}$).

The current multiplet of the stress tensor reads in particular

$$\mathcal{T}_0 = \frac{1}{2}\mathcal{J}_0^S, \qquad \mathcal{T}_1 = -\mathcal{A}_1^F + \mathcal{A}_1^S, \qquad \mathcal{T}_2 = 8\mathcal{J}_2^V - 2\mathcal{J}_2^F + \mathcal{J}_2^S.$$

It contains also a spin-1 current $\mathcal{T}_1$ (an R-current) and a spin-0 mass operator $\mathcal{T}_0$.

One can proceed similarly for the hypermultiplet and then combine the two in an N=2 finite theory, which is the case we are interested in here. This family is parametrized by a coupling constant g as well as the rank N_c of the gauge group, which we assume to be $SU(N_c)$. Multiplets having different minimal spins are orthogonal, but some pairs of multiplets have the same minimal spin. These, in general, mix under renormalization. In particular, there is a multiplet $\mathcal{T}^*$ mixing with $\mathcal{T}$.

At g different from zero the higher-spin currents acquire anomalous dimensions (and are extended to include other supersymmetric partners that disappear when $g = 0$). Let h_{2s} denote the minimal anomalous dimensions of the even-spin levels. The Ferrara–Gatto–Grillo–Nachtmann (FGGN) theorem states that, starting with the spin-2 level, the spectrum h_{2s} is positive, increasing and convex:

$$0 \leq h_{2s} \leq h_{2(s+1)}, \qquad h_{2(s+1)} - h_{2s} \leq h_{2s} - h_{2(s-1)}.$$

The most important implication of this theorem is that the OPE algebra generated by the multiplet of the stress tensor *does* close, in some special situation that we now describe.

We can classify conformal field theory in two classes:

i) *open* conformal field theory, when the quantum conformal algebra contains an infinite number of (generically non-conserved) currents;

ii) *closed* conformal field theory, when the quantum conformal algebra closes with a finite set of (conserved) currents.

The FGGN theorem implies in particular that the spectrum is identically zero if one h_{2s} is zero, and identically infinity if one h_{2s} is infinity. Precisely:
a) if $h_{2s} = 0$ for some $s > 1$, then $h_{2s} = 0$ $\forall s > 0$, and
b) if $h_{2s} = \infty$ for some $s > 1$, then $h_{2s} = \infty$ $\forall s > 1$.

Equipped with this, we can describe the moduli space of conformal field theory as a ball centred in free-field theory. As a radius r one can take the value of any h_{2s} with $s > 1$. The boundary sphere is the set of closed theories. The bulk is the set of open theories.

Let us discuss the two cases $r = 0$ and $r = \infty$ separately.

It is a rigorous and completely general consequence of the theorem that when $r = \infty$ all current multiplets have infinite anomalous dimensions and decouple from the OPE (with the only possible exception of $\mathcal{T}^*$, which is "screened" by $\mathcal{T}$). Supersymmetry plays an important role here, since each multiplet necessarily has some component with even spin, and therefore falls under the range of the Nachtmann theorem for $r \to \infty$.

The limit in which $r \to \infty$ is the limit of maximally strong interaction, in the sense that once the quantum conformal algebra closes, there is no way to make the interaction any stronger. It is not sufficient to take $g \to \infty$: in N=4 supersymmetric Yang–Mills theory, indeed, the $g \overleftrightarrow{1}/g$ duality suggests that the limit $g \to \infty$ at N_c fixed is free and not closed. To have the maximally strong interaction, one needs to take the large-N_c limit at the same time.

In the limit $r \to 0$ some currents with non-vanishing anomalous dimension might survive, in principle, since r is sensitive only to the *minimal* anomalous dimension of each even-spin level. It is nevertheless reasonable to expect that $r \to 0$ reduces to a free-field theory, and this is what we conjecture. Indeed, no interacting theory with infinitely many conserved currents is known. An interesting case, in this respect, is N=4 supersymmetric Yang–Mills theory, where the spectrum h_{2s} includes the full set of anomalous dimensions and therefore $r \to 0$ ensures that all higher-spin currents generated by the OPE are conserved.

The picture that has emerged can be summarized by the following statements.

i) Closed conformal field theory is the boundary of the moduli space of open conformal field theory.

ii) Closed conformal field theory is the exact solution to the strongly coupled large-N_c limit of open conformal field theory.

iii) A closed quantum conformal algebra determines uniquely the associated conformal field theory.

iv) A closed quantum conformal algebra is determined uniquely by two central charges, c and a.

We now comment on point (*iv*). The basic procedure to determine the quantum conformal algebra of closed conformal field theory (the so-called *fusion rules*) can be applied to any set of finite operators (for example, non-singlet currents with respect to some flavour group), although we focus on the minimal algebra (namely the one of the stress tensor) for the sake of generality. The procedure consists of the following steps. One studies the free-field OPE of an open conformal field theory and organizes the currents into orthogonal and mixing multiplets. Secondly, one turns a weak interaction on and computes the anomalous dimensions of the operators to the lowest orders in the perturbative expansion. Finally, one drops all the currents with a non-vanishing anomalous dimension. More generically, one can postulate a set of spin-0, 1 and 2 currents, which we call $\mathcal{T}_{0,1,2}$, and study the most general OPE algebra consistent with closure and unitary.

The closed N=2 quantum conformal algebra for generic c and a reads schematically

$$\begin{aligned}
\mathcal{T}_0\,\mathcal{T}_0 &= \frac{c}{|x|^4}+\frac{1}{|x|^2}\mathcal{T}_0,\\
\mathcal{T}_1\,\mathcal{T}_1 &= \frac{c}{|x|^6}+\frac{1}{|x|^4}\mathcal{T}_0+\left(1-\frac{a}{c}\right)\frac{1}{|x|^3}\mathcal{T}_1+\frac{1}{|x|^2}\mathcal{T}_2,\\
\mathcal{T}_2\,\mathcal{T}_2 &= \frac{c}{|x|^8}+\left(1-\frac{a}{c}\right)\frac{1}{|x|^6}\mathcal{T}_0+\left(1-\frac{a}{c}\right)\frac{1}{|x|^5}\mathcal{T}_1+\frac{1}{|x|^4}\mathcal{T}_2,
\end{aligned}$$

plus descendants and regular terms. We have emphasized those coefficients that are proportional to $(1-a/c)$. We observe that

∘ *the $c=a$ closed algebra is unique and coincides with the N=4 one.*

∘ *given c and a, there is a unique closed conformal algebra with N=2 supersymmetry.*

c has a natural interpretation as the central extension of the algebra, while the combination $(1-a/c)$ is a structure constant.

There might be a slightly more general, but still closed, structure, if the multiplet $\mathcal{T}^*$, which mixes with $\mathcal{T}$, does not drop. This algebra is determined by c, a and the anomalous dimension of $\mathcal{T}^*$.

Finally, we observe that in N=1 (and non-supersymmetric) theories the multiplet of the stress-tensor will not contain spin-0 partners, in general, but at most the R-current. The above considerations stop at the spin-2 and 1 levels of the OPE, but the procedure to determine the closed algebra is the same. What is more subtle is to identify the physical situation that the closed limit should describe.

3. Quantum irreversibility

Quantum field theory defines a natural fibre bundle. The base manifold is the space of physical correlators and the fibre is the space of scheme choices, with suitable regularity restrictions. A projection onto the base manifold is well defined and ensures scheme independence of the physical correlators. We call this bundle the *scheme bundle*.

The scheme bundle is equipped with a metric f and a fundamental one-form ω, defined as follows. Consider the two-point function of the trace Θ of the stress tensor. In four dimensions, we normalize it as

$$\langle \Theta(x)\ \Theta(0)\rangle = \frac{1}{15\pi^2}\frac{\beta^2(t)f(t)}{|x|^8}.$$

Reflection positivity ensures that $f\geq 0$. Actually, f is strictly positive throughout the RG flow, since the zeros of the two-point function are parametrized precisely by the factor β^2. Therefore f is a metric in the space of coupling constants, defined on the fibre. The beta function is also defined on the fibre, since it is scheme-dependent, but the combination $\beta^2(t)f(t)$ is scheme-independent and therefore lives on the base manifold. It is not a metric on the base manifold, however, since it vanishes at the critical points.

The fundamendal one-form ω is defined as

$$\omega = -\mathrm{d}\lambda\ \beta(\lambda)\ f(\lambda), \tag{7}$$

λ denoting the coupling constant, such that $\Theta = \beta(\lambda)\mathcal{O}$ for a suitable operator $\mathcal{O}$. In particular, $\lambda = \ln\alpha$ in a gauge field theory, where $\mathcal{O} = F^2/4$.

The central charge a multiplies the Gauss–Bonnet integrand, or Euler density,

$$\mathrm{G}_n = (-1)^{\frac{n}{2}}\varepsilon_{\mu_1\nu_1\cdots\mu_{\frac{n}{2}}\nu_{\frac{n}{2}}}\varepsilon^{\alpha_1\beta_1\cdots\alpha_{\frac{n}{2}}\beta_{\frac{n}{2}}}\prod_{i=1}^{\frac{n}{2}} R_{\alpha_i\beta_i}^{\mu_i\nu_i},$$

in the trace anomaly coupled to an external gravitational field. G_n is a non-trivial total derivative, i.e. the total derivative of a non-gauge-covariant current (the Chern–Simons form) and so it is defined up to trivial total derivatives, the divergence of a gauge-covariant current. The topological numbers calculated with a modified Gauss–Bonnet integrand of the form $\tilde{\mathrm{G}}_n = \mathrm{G}_n + \nabla_\alpha J^\alpha$ are exactly the same as those computed with G_n.

The modified integrand can be chosen to be linear in the conformal factor, and in that case we call it *pondered* Euler density. In particular $\tilde{\mathrm{G}}_n \propto \Box^{\frac{n}{2}}\phi$ for a conformally-flat metric $g_{\mu\nu} = \delta_{\mu\nu}\mathrm{e}^{2\phi}$. Writing $\sqrt{g}\mathrm{G}_n = \partial_\alpha C^\alpha$, where C^α is the Chern–Simons form, the *pondered* Chern–Simons form reads $\tilde{C}^\alpha = C^\alpha + \sqrt{g}J^\alpha$ and $\sqrt{g}\tilde{\mathrm{G}}_n = \partial_\alpha\tilde{C}^\alpha$.

In four dimensions, $\tilde{\mathrm{G}}_4$ reads $\tilde{\mathrm{G}}_4 = \mathrm{G}_4 - \frac{8}{3}\Box R = \mathrm{G}_4 + \nabla_\alpha J_4^\alpha$, with $J_4^\alpha = -\frac{8}{3}\nabla_\alpha R$. In six dimensions we have $\tilde{\mathrm{G}}_6 \equiv \mathrm{G}_6 + \nabla_\alpha J_6^\alpha$ with

$$J_6^\alpha = -\frac{48}{5}R^{\alpha\mu}\nabla_\mu R + \frac{102}{25}\nabla^\alpha R^2 - 12\nabla^\alpha(R_{\mu\nu}R^{\mu\nu}) - \frac{24}{5}\nabla^\alpha\Box R,$$

so that on conformally-flat metrics $\sqrt{g}\,\tilde{\mathrm{G}}_4 = 16\Box^2\phi$ and $\sqrt{g}\,\tilde{\mathrm{G}}_6 = 48\,\Box^3\phi$.

In generic n the pondered Euler density has the form

$$\tilde{\mathrm{G}}_n = \mathrm{G}_n + \nabla_\alpha J_n^\alpha = \mathrm{G}_n + \cdots + p_n\Box^{n/2-1}R, \qquad J_n^\alpha = \cdots + p_n\nabla^\alpha\Box^{\frac{n}{2}-2}R, \tag{8}$$

and on conformally-flat metrics $\sqrt{g}\tilde{\mathrm{G}}_n = -2(n-1)p_n\Box^n\phi$. Only the coefficient p_n in (8) is relevant for us and the definition of $\tilde{\mathrm{G}}_n$ makes it easily calculable.

The Euler characteristic of the n-dimensional sphere S^n is equal to 2. In our notation we can write

$$(-1)^{\frac{n}{2}}2^{\frac{3n}{2}+1}\pi^{\frac{n}{2}}\left(\frac{n}{2}\right)! = \int_{S^n}\sqrt{g}\mathrm{G}_n\,\mathrm{d}^n x = \int_{S^n}\sqrt{g}\tilde{\mathrm{G}}_n\,\mathrm{d}^n x = -2(n-1)p_n\int_{S^n}\Box^{\frac{n}{2}}\phi\,\mathrm{d}^n x.$$

The calculation in the sphere with metric $\mathrm{d}s^2 = \frac{\mathrm{d}x^2}{(1+x^2)^2}$, gives $p_n = -\frac{2^{\frac{n}{2}}n}{2(n-1)}$, which agrees with the known values in $n=4$ and $n=6$. In [7] the expression of $\tilde{\mathrm{G}}_8$ is also worked out and p_8 is checked.

Summarizing, on conformally flat metrics

$$\sqrt{g}\tilde{\mathrm{G}}_n = 2^{\frac{n}{2}}n\,\Box^{\frac{n}{2}}\phi.$$

According to ref. [6] the Euler density that should appear in the trace anomaly should be precisely the pondered Euler density, thereby removing the ambiguities associated with the coefficients a' of the trivial total derivative terms of the form $\nabla_\alpha J_n^\alpha$. The dependence of the trace anomaly on the conformal factor ϕ becomes extraordinarily simple:

$$\Theta = a_n\,\tilde{\mathrm{G}}_n + \text{conf. invs.} = 2^{\frac{n}{2}}n\,a_n\,\mathrm{e}^{-n\phi}\Box^{\frac{n}{2}}\phi, \tag{9}$$

and the relation between the total a-flow and the Θ two-point function becomes manifest. Normalizing a as in (9), the two-point function reads at criticality

$$\langle\Theta(x)\,\Theta(y)\rangle = -2^{\frac{n}{2}}n\,a_n\,\Box^{\frac{n}{2}}\delta(x-y)$$

and the expression for the a-flow is therefore:

$$a_n^{\mathrm{UV}} - a_n^{\mathrm{IR}} = \frac{\int \mathrm{d}^n x\,|x|^n\,\langle\Theta(x)\,\Theta(0)\rangle}{2^{\frac{3n}{2}-1}\,n\,\Gamma(n+1)}. \tag{10}$$

A convenient normalization of a_n is that it be equal to 1 for a real scalar field, and reads in general

$$N_s + f_n N_f + v_n N_v \tag{11}$$

for free-field theories with N_s real scalar fields, N_f Dirac fermions and N_v vectors. In $n = 4$ we change the normalization of a according to this convention ($f_4 = 11$, $v_4 = 62$) and write

$$\Delta a = a_{\mathrm{UV}} - a_{\mathrm{IR}} = \int_{\mathrm{UV}}^{\mathrm{IR}} \omega \geq 0. \tag{12}$$

The total RG flow of a is the integral of the fundamental one-form ω between the fixed points. Quantum irreversibility is measured by the invariant (i.e. scheme-independent) area of the graph of the beta function between the fixed points. In this integral, scheme independence is reparametrization invariance.

This formula can be checked to the fourth-loop order in the most general renormalizable theory. Here we focus on QCD in the conformal window in the neighbourhood of the asymptotic freedom point $N_f = \frac{11}{2} N_c$. The strategy for computing higher-loop corrections to the trace anomaly was developed in refs. [18]. See [7] for its extension to six dimensions.

Collecting the results of these references in a general formula, the third-loop RG flow of a reads

$$a_{\mathrm{UV}} - a_{\mathrm{IR}} = \frac{1}{2} f_{\mathrm{UV}} \beta_2 \alpha_{\mathrm{IR}}^2 + \mathcal{O}(\alpha_{\mathrm{IR}}^3),$$

where β_1 and β_2 are the first two coefficients of the beta function, $\beta(\alpha) = \beta_1 \alpha + \beta_2 \alpha^2 + \mathcal{O}(\alpha^3)$. Formula (12) gives exactly the same result. Concretely, with N_f flavours and N_c colours we have

$$a_{\mathrm{UV}} - a_{\mathrm{IR}} = \frac{44}{5} N_c N_f \left(1 - \frac{11}{2} \frac{N_c}{N_f}\right)^2.$$

In supersymmetric QCD the prediction can be compared with the exact formula (6). The check can be extended to the fourth-loop order [6], in both the supersymmetric and non-supersymmetric cases. In six dimensions, formula (10) has been checked to the fourth-loop order in the theory φ^3 [7].

I now discuss the extension of these results to all orders. Renormalization can be seen [6] as the restoration of positivity (or, better, boundedness from below) of the generating functional of 1PI diagrams in the Euclidean framework. This positivity is in general violated by the regularization procedure and divergences.

Here, we consider the induced action for the conformal factor ϕ. Despite the fact that ϕ is an external source, the positivity property holds because ϕ couples to Θ, an evanescent operator. At most, we might have to adjust the unique free parameter ("coupling constant") at our disposal: the a' ambiguity.

The quantum-irreversibility formula is derivable from the statement:
the induced effective action S_{R} for the conformal factor ϕ [19] is positive-definite throughout the RG flow, if and only if it is positive-definite at a given energy,
which implies the "a-theorem":

i) *a is non-negative;*

ii) *the total RG flow of a is non-negative and equal to the invariant area of the beta function:*

$$a_{\mathrm{UV}} - a_{\mathrm{IR}} = -\int_{\lambda_{\mathrm{UV}}}^{\lambda_{\mathrm{IR}}} \mathrm{d}\lambda \; \beta(\lambda) f(\lambda) \geq 0.$$

Now, $S_{\mathrm{R}}[\phi]$ is the solution of the equation $\Theta = \mathrm{e}^{-4\phi} \frac{\delta S_{\mathrm{R}}[\phi]}{\delta \phi}$. At criticality in the Euclidean framework we have

$$\Theta = \frac{1}{90(4\pi)^2} \left[a_* \mathrm{e}^{-4\phi} \Box^2 \phi + \frac{1}{6}(a_* - a'_*) \Box R \right]$$

and therefore

$$S_{\mathrm{R}}[\phi] = \frac{1}{180} \frac{1}{(4\pi)^2} \int \mathrm{d}^4 x \left\{ a_* (\Box \phi)^2 - (a_* - a'_*) \left[\Box \phi + (\partial_\mu \phi)^2 \right]^2 \right\}.$$

The two terms of $S_{\rm R}[\phi]$ have to be separately positive. In particular, positivity of the first term implies $a_* > 0$ in the IR if $a_* > 0$ in the UV. This is true, since $a_{\rm free} > 0$ in a free-field theory.

The quantity a' is defined up to an additive, coupling-independent constant and needs to be normalized at a given energy scale. The quantity whose RG flow is given by (12) is precisely a' and our statement amounts to showing that $\Delta a'$, which is certainly non-negative, is equal to Δa.

The second term of $S_{\rm R}[\phi]$ is positive at criticality if $a'_* \geq a_*$. This condition has to hold throughout the renormalization group flow, in particular $a'_{\rm UV} \geq a_{\rm UV}$ if and only if $a'_{\rm IR} \geq a_{\rm IR}$. Now, we know that $a'_{\rm UV} \geq a'_{\rm IR}$. Let us fix a' by demanding that a and a' coincide in the UV, $a'_{\rm UV} = a_{\rm UV}$. Then we have, combining the various inequalities derived so far, $a_{\rm UV} = a'_{\rm UV} \geq a'_{\rm IR} \geq a_{\rm IR}$ from which the claimed inequality $a_{\rm UV} \geq a_{\rm IR}$ follows.

Now, let us tentatively suppose that with the normalization $a'_{\rm UV} = a_{\rm UV}$ we have the strict inequality $a'_{\rm IR} > a_{\rm IR}$. We prove that this is absurd and conclude that $a'_{\rm IR} = a_{\rm IR}$.

We can do this by changing the normalization of a' with the shift $a' \rightarrow a'^{\,\rm new} = a' - a'_{\rm IR} + a_{\rm IR}$, so that $a'^{\,\rm new}_{\rm IR} = a_{\rm IR}$. We have $a'_{\rm UV} \rightarrow a'^{\,\rm new}_{\rm UV} = a'_{\rm UV} - a'_{\rm IR} + a_{\rm IR}$ and therefore $a'_{\rm UV}$ no longer satisfies the inequality $a'_* \geq a_*$, since $a'^{\,\rm new}_{\rm UV} < a_{\rm UV}$. This is a contradiction. We conclude that $a'_{\rm UV} = a_{\rm UV}$ if and only if $a'_{\rm IR} = a_{\rm IR}$.

These arguments are somewhat orthogonal, or complementary, to the approach *à la* spectral representation of [20]. In particular, knowledge about the (positivity) properties of the local parts of the correlators is of fundamental importance.

An extension of these ideas to odd dimensions, which is not straightforward since there is no trace anomaly in external gravity in odd dimensions, can be obtained by dimensional continuation. The resulting formula is testable, in principle, in models interpolating between pairs of free-field fixed points.

Finally, I stress that the phenomenon of quantum irreversibility is proper to the dynamical scale μ, i.e. it is the intrinsic drift of the renormalization group. Explicit scales (masses, Newton's constant and other dimensionful parameters) need not be described by formula (12). Moreover, formula (12) has to be replaced by a more complicated expression also when the stress tensor is not truly finite, but mixes with other operators (a well-known example is the $\lambda\varphi^4$-theory - see sect. 2.3 of [6] for details). These are signals of the richness of higher-dimensional conformal field theories with respect to the two-dimensional ones. There is, nevertheless, a remarkable subset, the set $c = a$, where the two-dimensional properties are best reproduced. This set admits a generalization to arbitrary dimension and it is defined as the set of conformal field theories having a trace anomaly quadratic in the Ricci tensor and Ricci curvature [8].

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