On the Holographic S–matrix

I.Ya.Aref'eva
Steklov Mathematical Institute, Russian Academy of Sciences
Gubkin St.8, GSP-1, 117966, Moscow, Russia

Abstract

The recent proposal by Polchinski and Susskind for the holographic flat space S–matrix is discussed. By using Feynman diagrams we argue that in principle all the information about the S–matrix in the interacting field theory in the bulk of the anti-de Sitter space is encoded into the data on the timelike boundary. The problem of locality of interpolating field is discussed and it is suggested that the interpolating field lives in a quantum Boltzmannian Hilbert space.

According to the holographic principle [1, 2] one should describe a field theory on a manifold M which includes gravity by a theory which lives on the boundary of M. Two prominent examples of the holography are the Matrix theory [3] and the AdS/CFT correspondence [4, 5, 6]. The relation between quantum gravity in the anti-de Sitter space and the gauge theory on the boundary could be useful for better understanding of both theories. In principle CFT might teach us about quantum gravity in the bulk of AdS. Correlation functions in the Euclidean formulation are the subject of intensive study (see for example [7]-[15]). The AdS/CFT correspondence in the Lorentz formulation is considered in [16]-[23].

Recently Polchinski [20] and Susskind [21] have proposed an expression for the S–matrix in flat spacetime in terms of the large N limit of the gauge theory living on the boundary of the AdS space. A related derivation is given in [22]. In this note we discuss this proposal.

It was suggested [16, 17] that a quantum field $\Phi(t, \mathbf{x})$ in the bulk of AdS_5 and the corresponding operator $\mathcal{O}(t, \hat{\mathbf{x}})$ in the gauge theory at the boundary are related by a simple formula

$$\lim_{r\to\infty} r^4 \Phi(t, \mathbf{x}) = \mathcal{O}(t, \hat{\mathbf{x}}) \tag{1}$$

Such a formula might be interpreted from two different points of view. One can assume that the field $\mathcal{O}(t, \hat{\mathbf{x}})$ in CFT is known and study what could be a field $\Phi(t, \mathbf{x})$ in the bulk. Or one can start from a field in the bulk of AdS and study its limiting behaviour at the boundary. In this note we are interested in the discussion of the latter approach which perhaps can be helpful for a clarification of the holographic principle.

We consider the following question. Let us take a simple model of QFT in the flat 5–dimensional spacetime, say the massless scalar field with Φ^3 interaction. Can we get the ordinary S–matrix for this model if we start from a quantum field theory in the bulk of AdS and use the data on the time like boundary of AdS in the flat space limit? We will see that an important point here is to answer to the question what do we mean under the quantum field theory in the bulk of AdS? In other words, how to quantize the Φ^3 theory in AdS if we want to reproduce the flat space S-matrix by using only "holographical" data on the timelike infinity of AdS?

First we remind the standard formulae of the scattering theory in the 5–dimensional Minkowski spacetime. There are four different formulations of S-matrix on the flat spacetime, see for example [24, 25, 26]. They include the *in-out* formalism, the LSZ formula, the Feynman diagrams and the on-shell functional integral representation. It is well known that all of them are equivalent in the flat spacetime. However in a curve background they could lead to different prescriptions.

We consider a massless scalar field with the action

$$I = \int d^5x \left\{ -\frac{1}{2} (\partial_\mu \Phi)^2 - V(\Phi) \right\} \tag{2}$$

where $V(\phi) = \lambda_1 \phi^3 + \lambda_2 \phi^4 + \dots$. In the *in-out* formalism one defines an interpolating field

$$a(\mathbf{k}, x^0) = i \int d^4x (F_{\mathbf{k}} \partial_0 \Phi - \partial_0 F_{\mathbf{k}} \Phi) \tag{3}$$

and in–out annihilation operators

$$a_{out,in}(\mathbf{k}) = \lim_{x^0\to\pm\infty} a(\mathbf{k}, x^0) \tag{4}$$

Here $F_{\mathbf{k}}$ is the solution of the wave equation with a positive frequency,

$$\Box F_{\mathbf{k}} = 0 \ ,$$

$$F_{\mathbf{k}}(x) = e^{i\omega(\mathbf{k})x^0 - i\mathbf{k}\mathbf{x}}/(2\pi)^2\sqrt{2\omega(\mathbf{k})} \ , \quad \omega(\mathbf{k}) = |\mathbf{k}|$$

S–matrix is

$$\langle \mathbf{p}_1, \ldots, \mathbf{p}_m; out|\mathbf{k}_1, \ldots, \mathbf{k}_n; in\rangle = \langle a_{out}(\mathbf{p}_1)\ldots a_{out}(\mathbf{p}_m)a^*_{in}(\mathbf{k}_1)\ldots a^*_{in}(\mathbf{k}_n)\rangle \tag{5}$$

One can prove that (5) can be written in the following standard form (the LSZ formula)

$$\langle a_{out}(\mathbf{p}_1)\ldots a^*_{in}(\mathbf{k}_n)\rangle = \int \prod_{i=1}^{m} F_{\mathbf{p}_i}(x_i)d^5x_i \prod_{j=1}^{n} F^*_{\mathbf{k}_j}(y_j)d^5y_j \Box_{x_1}\ldots\Box_{y_n} G(x_1, \ldots, y_n) + ... \tag{6}$$

Here G is the Feynman Green function,

$$G(x_1, \ldots, x_n) = \langle T(\Phi(x_1)\ldots\Phi(x_n))\rangle \tag{7}$$

This can be can be represented in the perturbation theory as

$$\langle T(\Phi(x_1)\ldots\Phi(x_n))\rangle = \langle T(\Phi_0(x_1)\ldots\Phi_0(x_n)S)S^*\rangle \tag{8}$$

where

$$S = T\exp\left[-i\int d^5x \ V(\Phi_0)\right] \tag{9}$$

and $\Phi_0(x)$ is the free field. It is the form (6)–(9) that is usually used for computations in perturbation theory and it can be expressed by using the Feynman diagrams.

Now let us consider this quantum field theory in the AdS_5 spacetime. The universal anti–de Sitter space is conformal to one half of the Einstein static cylinder. The metric is

$$ds^2 = R^2\cosh^2\chi[-d\tau^2 + d\sigma^2 + \sin^2\sigma d\Omega] \tag{10}$$

where $\sigma = 2\arctan e^{\chi} - \frac{1}{2}\pi$ and the Penrose diagram is shown on Fig. 1 There are null geodesics and $\mathcal{T}$ is time like infinity.

The action for the scalar field in AdS_5 is

$$I = \int d^5x\sqrt{g}\left\{-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_v\Phi - V(\Phi)\right\} \tag{11}$$

Now we have to quantize the field and try to repeat the main steps of the scattering theory (4)–(7). Moreover we want to study the holographic principle and therefore we shall try to encode the data on the scattering matrix into a theory on the timelike infinity $\mathcal{T}$.

One of problems we immediately get in the attempt to build a quantum theory in AdS space is that there exists no Cauchy surface in the space. It is clear from Fig. 1 that for any spacelike surface (for example a horizontal section on Fig. 1) one can find null geodesics which never intersect the surface. So we have to restrict ourself to build a quantum theory only in a region of AdS space. This might be enough since our goal is to study the flat space limit.

AdS behaves like a cavity with reflecting walls [21] and there are no ordinary scattering states in AdS space. It was proposed in [20, 21] to introduce sources and detectors on the boundary. Then in the flat limit one can get the flat spacetime scattering amplitude if one holds the external proper momenta fixed as $R \to \infty$. One takes the AdS metric in the form

$$\begin{aligned} ds^2 &= R^2[-(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2d\Omega_3^2] \\ &= R^2[-(1+r^2)dt^2 + d\mathbf{x}d\mathbf{x} - \frac{(\mathbf{x}d\mathbf{x})^2}{1+r^2}] \end{aligned} \tag{12}$$

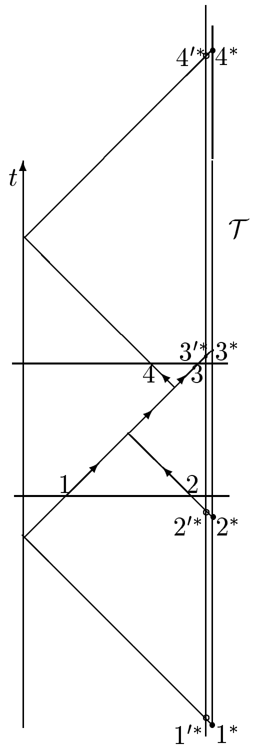

Figure 1: Born approximation for pre-scattering in AdS

where $r^2 = \mathbf{x}\mathbf{x}$. One obtains the flat spacetime metric at $R \to \infty$ in the "proper" coordinates $T = Rt, \mathbf{X} = \mathbf{x}R$. One denotes $\mathbf{x} = r\hat{\mathbf{x}},\ \hat{\mathbf{x}} \in S^3$ and for momenta $\mathbf{p} = |\mathbf{p}|\hat{\mathbf{p}}$.

Polchinski [20] gave the following prescription for the flat space S-matrix

$$\langle \mathbf{p}_1, \ldots, \mathbf{p}_m; out|\mathbf{k}_1, \ldots, \mathbf{k}_n; in\rangle = \lim_{R\to\infty} Z^{-1}\langle \prod_i \alpha_-(R|\mathbf{p}_i|, \hat{\mathbf{p}}_i) \prod_j \alpha_+(R|\mathbf{k}_j|, \hat{\mathbf{k}}_j)\rangle \tag{13}$$

Here

$$\alpha_\pm(\omega, \hat{\mathbf{p}}) = \lim_{r\to\infty} \int_{\Sigma_r} d\sigma^\mu (F^{(\pm)}_{\omega\hat{\mathbf{p}}}\partial_\mu\Phi - \partial_\mu F^{(\pm)}_{\omega\hat{\mathbf{p}}}\Phi) = \int dt d^3\hat{x} f^{(\pm)}_{\omega\hat{\mathbf{p}}}(t, \hat{\mathbf{x}})\phi(t, \hat{\mathbf{x}})\ . \tag{14}$$

Functions $F^{(\pm)}_{\omega\hat{\mathbf{p}}}(t, \mathbf{x})$ are classical solutions of the wave equation $\Box F^{(\pm)}_{\omega\hat{\mathbf{p}}} = 0$ with the following properties. In the neighborhood of the origin $r = 0$

$$F^{(\pm)}_{\omega\hat{\mathbf{p}}}(t, x) \approx \psi^{(\pm)}_{\omega\hat{\mathbf{p}}}(t, \mathbf{x}) e^{-i\omega(t - \hat{\mathbf{p}}\hat{\mathbf{x}})} \tag{15}$$

where $\psi^{(\pm)}_{\omega\hat{\mathbf{p}}}(t, \mathbf{x})$ goes to identity as $R \to \infty$. So in the flat space limit one gets the plane wave solution. The asymptotic behaviour at the boundary is

$$\lim_{r\to\infty} F^{(\pm)}_{\omega\hat{\mathbf{p}}}(t, \mathbf{x}) = f^{(\pm)}_{\omega\hat{\mathbf{p}}}(t, \hat{\mathbf{x}}) = G^{(\pm)}_\omega\left(t \mp \frac{\pi}{2}, |\hat{\mathbf{x}} \mp \hat{\mathbf{p}}|\right) e^{-i\omega t} \tag{16}$$

where

$$G^{(\pm)}_\omega(\tau, \theta) = -e^{\pm i\pi\omega/2}\left(\frac{2}{\omega}\right)^{3/2} \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{\omega}{2}(\theta^2 + \tau^2)\right\} \tag{17}$$

The factor Z is

$$Z = \int dt d^4x \prod_i \psi^{\pm}_{\omega_i\hat{\mathbf{p}}_i}(t, \mathbf{x}) \tag{18}$$

The operator $\phi(t, \hat{\mathbf{x}})$ is defined by the relation

$$\lim_{r \to \infty} r^4 \Phi(t, \mathbf{x}) = \phi(t, \hat{\mathbf{x}}) \tag{19}$$

and it is interpreted as an operator $\mathcal{O}$ in the gauge theory on S^3.

For quantization one needs a complete orthonormal system of solutions of the wave equation. It seems functions $\{F_{\omega \hat{\mathbf{p}}}\}$ do not form such a system. Quantization of the free scalar field in AdS has been performed in [27]. In the coordinates $\sinh\chi = \tan\rho$ the mode expansion of scalar field is

$$\Phi(t, \rho, \Omega) = \sum_{m,n,l} [\frac{(l+n+2)(l+n+3)}{(n+1)(n+2)C_l}]^{1/2} \sin^l \rho \cos^4 \rho P_n^{(l+1,2)}(\cos 2\rho)$$

$$[e^{-i\omega_{ln} t} Y_{lm}(\Omega) a_{mnl} + h.c.]$$

Creation and annihilation operators a^*_{mnl}, a_{mnl} satisfy the standard Bose commutation relations. Let us notice here that the Hibert space of the large N limit is not the ordinary Bose Fock space, but rather it is the quantum Boltzmann space [28, 29].The inteprolating field is nonlocal and it should live int the quantum Boltzmannian Hilbert space. It seems that entangled commutation relations [30] are appropriate for describing collective degrees of freedom.

The on-shell functional integral representation for S-matrix [26] reads

$$S(\Phi_{in,out}) = \int e^{iI(\Phi)} \mathcal{D}\Phi$$

One integrates over the field configurations with the prescribed behaviour at infinity.

It will be convenient to think on the S–matrix in the first quantized interpretation. Let us for simplicity consider a process $2 \to 2$ in the Born approximation . The answer is

$$\langle p_1 p_2 | S | p'_1 p'_2 \rangle = \int \prod_{i=1}^{2} \frac{e^{i\omega(\vec{p}_i)t - i\vec{x}\vec{p}_i}}{(2\pi)^2)\sqrt{2\omega(p_i)}} \cdot \prod_{j=1}^{2} \frac{e^{-i\omega(p'_j)t + i\vec{y}\vec{p}_j}}{(2\pi)^2 \sqrt{2\omega(p_j)}} \cdot K(x, y) dx dy \tag{20}$$

Here we interpret the plane wave as the amplitude of the probability to find a particle with given momenta and energy at the point x. The kernel $K(x, y)$ is the transition amplitude from the point x to the point y. The plane wave describes the travelling of free particle along the geodesics. We consider the pre-S-matrix $< ...|\mathcal{S}(R)|... >$ using an intuitive first quantized approach. We take $\mathcal{S}(R)$ in the form which naturally generalizes the Feynman flat space formula (20),

$$\langle \omega'_1 \hat{p}'_1, \omega'_2 \hat{p}'_2, \ldots, \omega'_m \hat{p}'_m | \mathcal{S}(R) | \omega_1 \hat{p}_1, \ldots, \omega_n \hat{p}_n \rangle = \tag{21}$$

$$\int \prod_i \sqrt{g(x_i, t_i)} dx_i dt_i \; K_{as}(\omega'_i \hat{p}'_i | x_i, t_i) K_{as}(x_i, t_i | \omega_{j_i} \hat{p}_{j_i}) +$$

$$\int \prod_i \sqrt{g(x_i, t_i)} dx_i dt_i K_{as}(x_i, t_i | \omega_{j_i} \hat{p}_{j_i}) \prod_j \sqrt{g(x_i, t_i)} dx_i dt_i \; K_{as}(\omega'_i \hat{p}'_i | x'_i, t'_i) K(x'_1, t'_1, ...x_n, t_n)$$

The kernel admits the first quantized representation. In particular, for $2 \to 2$ process the formula (21) becomes

$$\langle \omega'_1 \hat{p}'_1, \omega'_2 \hat{p}'_2, | \mathcal{S}(R) | \omega_1 \hat{p}_1, \omega_2 \hat{p}_2 \rangle = \tag{22}$$

$$\int \sqrt{g(x_1, t_1)} dx_1 dt_1 \sqrt{g(x_2, t_2)} dx_2 dt_2 \sqrt{g(x'_1, t'_1)} dx'_1 dt'_1 \sqrt{g(x'_2, t'_2)} dx'_2 dt'_2 \; K_{as}(\omega'_1 \hat{p}'_1 | x'_1, t'_1) \cdot$$

$$K_{as}(\omega'_2 \hat{p}'_2 | x'_2, t'_2) K_{as}(x_1, t_1 | \omega_1 \hat{p}_1) K_{as}(x_2, t_2 | \omega_2 \hat{p}_2) [\delta(x_1 - x'_1) \delta(t_1 - t'_1) + ...] +$$

$$\sqrt{g(x,t)}dxdt\sqrt{g(x',t')}dx'dt'\ K_{as}(\omega_1'\hat{p}_1'|x',t')K_{as}(\omega_2'\hat{p}_2'|x',t')\cdot$$
$$K_{as}(x,t|\omega_1\hat{p}_1)K_{as}(x,t|\omega_2\hat{p}_2)K(x',t';x,t)$$

Here $K(x,t;x',t')$ is the transition amplitude from the point x,t to point x',t' . The 2-point transition amplitude includes the sum over all possible curves with branches,

$$K(x,t;x',t') = \sum_{topol}\int_{x,t)}^{x',t')} e^{iS}\mathcal{D}x\mathcal{D}t \tag{23}$$

In the Born approximation one performs only summation over smooth curves

$$K_{Born}(x,t;x',t') = \int_{x,t)}^{x',t')} e^{iS}\mathcal{D}x\mathcal{D}t \tag{24}$$

For $K_{as}(\omega_1'\hat{p}_1'|x_1',t_1')$ one expects the formula

$$K_{as}(\omega\hat{p}|x,t) = K_{as}(y_{as}(\omega\hat{p}),t_{as}(\omega\hat{p})|x,t) \tag{25}$$

Intuitively it is clear that these asymptotic kernel are solutions of wave equations in the AdS space with special boundary conditions. ¿From the previous discussions it seems natural to take

$$K_{as}(\omega\hat{p}|x,t) = \frac{F^{(\pm)}_{\omega\hat{p}}(t,x)}{\psi^{\pm}_{\omega\hat{p}}(t,x)} \tag{26}$$

Therefore, for pre-$\mathcal{S}$-matrix we obtain

$$\langle\omega_1'\hat{p}_1',\omega_2'\hat{p}_2',\ldots,\omega_m'\hat{p}_m'|\mathcal{S}(R)|\omega_1\hat{p}_1,\ldots,\omega_n\hat{p}_n\rangle_{connected\ part} = \tag{27}$$

$$\int\prod_i \frac{F^{(\pm)}_{\omega_i\hat{p}_i}(t_i,x_i)}{\psi^{\pm}_{\omega_i\hat{p}_i}(t_i,x_i)}\sqrt{g(x_i,t_i)}dx_idt_i\prod_j\frac{F^{(\pm)}_{\omega_i'\hat{p}_i'}(t_i',x_i')}{\psi^{\pm}_{\omega_i'\hat{p}_i'}(t_i',x_i')}\sqrt{g(x_i',t_i')}dx_i'dt_i'K(x_1',t_1',...x_n,t_n)$$

In conclusion, in this note we have given a recipe of how to encode an information on the flat space S-matrix into the holographic data on the boundary of AdS. Although this recipe is not very transparent it permits to decode the flat space S-matrix. It would be interesting to study a relation between the recipe and the operator prescription for the flat space holographic S-matrix. We have proposed that the interpolating field lives in the Boltzmannian Hilbert space.

Acknowledgments

I would like to thank I.V.Volovich for discussions. This work was supported in part by INTAS grant 96-0698.

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