Pure Discrete Sector in D0-Branes

I.Ya. Aref'eva
Steklov Mathematical Institute, Russian Academy of Sciences
Gubkin St.8, GSP-1, 117966, Moscow, Russia
A.S. Koshelev
Physical Department, Moscow State University, Moscow, Russia, 119899
P. B. Medvedev
Institute of Theoretical and Experimental Physics,
B.Cheremushkinskaya st.25, Moscow, 117218

Abstract

We present the results of study the spectrum of $SU(2) \times SO(2)$ matrix supersymmetric quantum mechanics performed in hep-th/9911149. A new branch in the spectrum is found. It contains a pure discrete spectrum. These states are characterized by the constraint $\bar{Q}\Psi_{n_q} = 0$, where n_q is the maximum value of a quantum number n (the Lorentz angular momentum) in a given energy level.

1 Introduction

In this talk we present the results obtained in our paper [1], where we study the spectrum of $SU(2)\times SO(2)$ matrix supersymmetric quantum mechanics. It is described by the Hamiltonian $H = H_B + H_F$:

$$H = \frac{g_s}{2}(\pi_1^2 + \pi_2^2) + \frac{1}{2g_s}|\varphi_1 \times \varphi_2|^2 - \frac{i}{2}\varepsilon^{abc}(\varphi_1^a + i\varphi_2^a)\chi^b\chi^c - \frac{i}{2}\varepsilon^{abc}(\varphi_1^a - i\varphi_2^a)\bar{\chi}^b\bar{\chi}^c \tag{1}$$

and constrained by the Gauss law $j^a|\Psi\rangle = 0$. π_i^a, φ_j^b and $\bar{\chi}^a$, χ^b are canonically conjugated isovector pairs with $[\pi_i^a, \varphi_j^b] = -i\delta^{ab}\delta_{ij}$, $\{\bar{\chi}^a\chi^b\} = \delta^{ab}$. The Hamiltonian (1) is invariant under $SU(2)$ rotations and redundant Lorentz rotations generated by

$$j^a = l^a + s^a = \varepsilon^{abc}\varphi_i^b\pi_i^c - i\varepsilon^{abc}\bar{\chi}^b\chi^c \text{ and } N = N_B + N_F = \varphi_1^a\pi_2^a - \varphi_2^a\pi_1^a + \frac{1}{2}\bar{\chi}^a\chi^a. \tag{2}$$

There are two supercharges $\{Q, \bar{Q}\} = H$. The supercharges commute with the Hamiltonian up to the generator of $SU(2)$ rotations vanishing on the physical states.

This model has attracted a lot of attention in the last two decades (see refs. in [1]). From the previous investigations we know the following about the spectrum of the model.

- There is a continuous spectrum starting from zero [2].
- There is no a normalized zero mode [3].

This character of spectrum has lead to the commonly accepted belief about the instability of the model. Note, that in the literature there is no a statement about the whole spectrum. In fact there is another branch in the spectrum, that does not start from zero and that does not contain a continuous spectrum [1].

The branch that contains the continuous spectrum is nothing but a supersymmetric generalization of the pure bosonic model. The classical particle can escape to infinity along the bottom of the potential valleys but quantum fluctuations close the flat directions by confining potentials. In this case wave functions fall off rapidly and the spectrum (s-wave part of H_B spectrum) is purely discrete [4]-[6]. In the supersymmetric case there is an additional negative contribution to the effective potential coming from the Yukava terms in H_F which cancel the bosonic one, so the flat directions became open. As a result of this cancellation wave functions are no more confined and the spectrum becomes continuous [2]. In bosonic case we have stability and we loose it in supersymmetric case.

A new pure discrete branch does not have a counterpart in the bosonic model, it's states correspond to the p-wave ($l = 1$) part of the H_B spectrum. These wave functions have unusual properties, namely they are not located at the minimum of the classical potential $|\varphi_1 \times \varphi_2|^2$. Moreover, in the new branch wave functions are zero at the bottom of the valleys.

Technically, to separate between the branches we have to specify on which states the above mentioned boson-fermion cancellation takes place. It follows from the symmetries of H that the energy levels are four-fold degenerate and it is convenient to arrange the states into quartets enumerated by a number n_q. Upon quantization n_q, the eigenvalues of N can be integer or half-integer. Our analysis indicates the boson-fermion cancellation in the sector with even n_q and no cancellation for half-integer n_q (states with odd n_q are forbidden). As a result, there is only a discrete spectrum in the half-integer n_q sector and the model is stable. The lowest energy in this sector is positive, so the supersymmetry is broken in this sector.

Our results are depicted in the Table (Fig.1). States from the quartets are situated along the horizontal lines. Eigenvalues of N for the states in a quartet n_q are $n_q, n_q - 1/2$ and $-n_q + 2, -n_q + 3/2$. Each item of the quartet can be obtained from any other one by the action of supercharges or the permutation operator P.

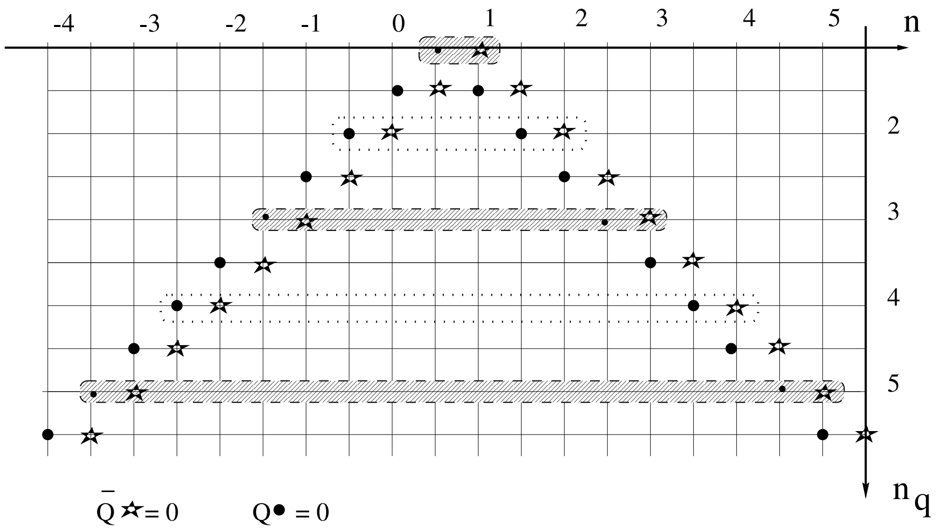

Figure 1: Structure of the spectrum: stars correspond to $\bar{Q}\Psi = 0$, filled circles correspond to $Q\Psi = 0$; domains bounded by dots correspond to the presence of the continuous spectrum; shared domains correspond to excluding values of n and n_q. Quartets are located along the horizontal lines.

Table 1 demonstrates that there is a possibility to put a superselection rule to exclude the continuous spectrum. Namely, taking any state from a quartet with half-integer n_q one gets with guarantee the state of the discrete spectrum. One can also specify the purely discrete sector by using the supercharges, i.e. the states with integer n satisfy the constraint $Q\Psi_n = 0$, meanwhile states with half-integer n satisfy $\bar{Q}\Psi_n = 0$.

Our main tool is the proper coordinates, four angles γ_1 ,γ_2, α, θ and two radii f and g. These coordinates have been already used in [13]. In these coordinates we will find an explicit solution to the Gauss law constrains and single out quantum number n. Due to this parametrisation the spectral problem for a supersymmetric version of the model with 6 degrees of freedom will be reduced to a pair of systems of partial differential equations in f and g. The choice of a system is dictated by the value of n_q. For half-integer n_q we have just one equation on one function of f and g. For even n_q we have three equations on three functions of f and g.

We present asymptotic solutions of these two systems deriving thereby the asymptotic formula for the spectrum. For half-integer n_q we deal with H_B for $l = 1$ and the character of the spectrum can be understood from the form of potential. Since we have a confining potential the spectrum is discrete. We present an asymptotic solution of the Schrödinger equation and corresponding formula for the spectrum. For even n_q we have a matrix second order differential operator. Just for this system a vanishing of the bosonic confining potential takes place on special states and the matrix differential operator has the continuous spectrum. To get it we find the asymptotic solution of the system of three equations. Besides the continuous spectrum this operator possesses the discrete spectrum.

2 Algebraic Structure of $SU(2)\times SO(2)$ Supersymmetric Quantum Mechanics

2.1 Energy Level Degeneration

Commutators of j^a and N with Q and $\bar{Q}$ are:

$$Q^2 = 0, \quad \bar{Q}^2=0, \quad [Q,j^a]=[\bar{Q},j^a]=0, \tag{3}$$

$$[N,Q] = -\frac{1}{2}Q, \quad [N,\bar{Q}]=\frac{1}{2}\bar{Q}, \quad [N,H]=0. \tag{4}$$

Note, that the algebra (3)-(4) is not affected by a shift of N by a constant. As compare with [11] our N is shifted on 1/4.

On the physical states ($j^a|\Psi\rangle=0$) there are three operators commuting with H: Q, $\bar{Q}$ and N. This provides the degeneration of energy levels. In this section we discuss the irreducible representations v_E of the algebra (3) - (4) for a non-zero energy E.

Lemma. Let $|\Psi\rangle\in v_E$ with $E\neq 0$ be an eigenvector of H and N, then $Q|\Psi\rangle=0$ or $\bar{Q}|\Psi\rangle=0$.

Proof. Suppose that $Q|\Psi\rangle\neq 0$ and $\bar{Q}|\Psi\rangle\neq 0$ then one has two additional eigenvectors of N with the same energy:

$$N(Q|\Psi\rangle)=(n-\frac{1}{2})(Q|\Psi\rangle), \quad N(\bar{Q}|\Psi\rangle)=(n+\frac{1}{2})(\bar{Q}|\Psi\rangle). \tag{5}$$

One can check, that the operator

$$K=HN+\frac{1}{4}[Q,\bar{Q}]$$

is one more Casimir of the algebra (3) - (4). By a simple calculation one finds:

$$K|\Psi\rangle = E(n+\frac{1}{4})|\Psi\rangle-\frac{1}{2}\bar{Q}Q|\Psi\rangle=E(n-\frac{1}{4})|\Psi\rangle+\frac{1}{2}Q\bar{Q}|\Psi\rangle, \tag{6}$$

$$K(Q|\Psi\rangle) = E(n-\frac{1}{4})(Q|\Psi\rangle), \tag{7}$$

$$K(\bar{Q}|\Psi\rangle) = E(n+\frac{1}{4})(\bar{Q}|\Psi\rangle). \tag{8}$$

For v_E to be irreducible the value of the Casimir on $|\Psi\rangle$ should be equal to its value on $Q|\Psi\rangle$ and $\bar{Q}|\Psi\rangle$. From equations (6)-(8) we see that this requires

$$Q|\Psi\rangle=0 \quad \text{or} \quad \bar{Q}|\Psi\rangle=0.$$

(remind that if $Q|\Psi\rangle=0$ then $|\Psi\rangle=Q|\chi\rangle$ with $|\chi\rangle=\frac{1}{E}\bar{Q}|\Psi\rangle$). Since $\bar{Q}$ and Q act as raising and lowering operators (eqs. (5)) one can introduce a label: $v_{E,n}$ where n is the value of N on the vector $|\Psi\rangle\in\ker\bar{Q}$.

$$\begin{tikzcd} \mathbf{0} & \Psi_{n-1/2} \arrow[l, "Q"'] \arrow[r, shift left, "\bar{Q}"] & \Psi_n \arrow[l, shift left, "Q"] \arrow[r, "\bar{Q}"] & \mathbf{0} \end{tikzcd}$$

So, we find that $v_{E,n}$ is a vector space spanned by two eigenvectors of N: $|\Psi_1\rangle$ and $|\Psi_2\rangle$ such that

$$\bar{Q}|\Psi_1\rangle=0=Q|\Psi_2\rangle \quad \text{and} \quad N|\Psi_1\rangle=n|\Psi_1\rangle, \quad N|\Psi_2\rangle=(n-\frac{1}{2})|\Psi_2\rangle.$$

In addition to the algebra (3),(4) the Hamiltonian (1) admits one more symmetry. The bosonic part of the Hamiltonian is invariant under the discrete transformation $\varphi_1\longleftrightarrow\varphi_2$. To extend this symmetry for the supersymmetric case we consider: $\chi\to\mu\bar{\chi}, \quad \bar{\chi}\to\nu\chi$. The invariance of the Hamiltonian and of the commutation relations gives the following restrictions on μ,ν: $-\mu^2=\nu^2=i, \quad \mu\nu=1$. One more restriction, $\nu = i\mu$ comes from imposing the condition for transformation law of supercharges $Q, \bar{Q}$ to be homogeneous.

The system of equations for μ and ν has a solution (up to an overall sign): $\mu = e^{-i\frac{\pi}{4}}, \quad \nu = e^{i\frac{\pi}{4}}$. Denote the operator of this symmetry by P, then we have

$$[P,H] = 0, \quad [P,j] = 0, \quad PQ - \mu\bar{Q}P = 0, \quad P\bar{Q} - \nu QP = 0, \quad \{P,N\} = \frac{3}{2}P, \quad P^2 = 1. \tag{9}$$

Note, that P acts on the spinor monomials without permutations of spinor factors, because these permutations are incompatible with the invariance of the anticommutation relations.

$$\begin{tikzcd}
n \arrow[r, leftrightarrow, "P"] \arrow[d, shift right, "Q"'] & -n+3/2 \arrow[d, shift right, "Q"'] \\
n-1/2 \arrow[r, leftrightarrow, "P"] \arrow[u, shift right, "\bar{Q}"'] & -n+2 \arrow[u, shift right, "\bar{Q}"']
\end{tikzcd}$$

Figure 2: States in a quartet

As $Pv_{E,n} = v_{E,2-n}$ **(action of P-operator on the states cab be easily calculated [1])** the irreducible representations of the enlarged algebra (3)-(4)+(9) are direct sums: $V_{E,n_q} = v_{E,n} + v_{E,2-n}$, where n_q is the maximal number of n and $2-n$. To avoid double-counting one has to restrict the range for n as to exclude $n < -n+2$ which gives $n > 1$. In the exceptional case $n = 1$: $v_{E,1} = v_{E,-1+2}$, *i.e.* for $n = 1$ one has a two-dimensional instead of a four-dimensional irreducible representation. The further specification of the range will be given below.

2.2 Gauss Law and $N|\Psi\rangle = n|\Psi\rangle$ in Components

A general wave function is a vector of the eight-dimensional fermionic Fock space based on the fermionic vacuum $|0\rangle$: $\chi^a|0\rangle = 0$. It is useful to arrange the fermionic states according to their parity as follows:

$$|\Psi\rangle = \left(\psi_0 + \psi^1\bar{\chi}^2\bar{\chi}^3 + \psi^2\bar{\chi}^3\bar{\chi}^1 + \psi^3\bar{\chi}^1\bar{\chi}^2 + \ \tilde{\psi}_0\bar{\chi}^1\bar{\chi}^2\bar{\chi}^3 + \tilde{\psi}^1\bar{\chi}^1 + \tilde{\psi}^2\bar{\chi}^2 + \tilde{\psi}^3\bar{\chi}^3\right)|0\rangle, \tag{10}$$

where ψ_0, ψ^a, $\tilde{\psi}_0$ and $\tilde{\psi}^a$ are some functions of $\vec{\varphi}_1$, $\vec{\varphi}_2$.

The Gauss law

$$j^a|\Psi\rangle = 0$$

fixes the $SU(2)$ transformation properties of the component wave functions ψ^a, $\tilde{\psi}^a$. These are given by:

$$l^a|\Psi\rangle = -s^a|\Psi\rangle.$$

The spin operator $\vec{s}$ acts on fermionic Fock states and its action can be easily calculated to give

$$[l^a, \psi_0] = 0, \quad [l^a, \psi^b] = i\epsilon^{abc}\psi^c, \quad [l^a, \tilde{\psi}_0] = 0, \quad [l^a, \tilde{\psi}^b] = i\epsilon^{abc}\tilde{\psi}^c. \tag{11}$$

Hence, we conclude that:

- ψ_0 and $\tilde{\psi}_0$ are $SU(2)$ singlets,
- $\vec{\psi} = (\psi^1, \psi^2, \psi^3)$ and $\vec{\tilde{\psi}} = (\tilde{\psi}^1, \tilde{\psi}^2, \tilde{\psi}^3)$ are $SU(2)$ triplets.

One can check that

$$\vec{l}^2\psi_0 = \vec{l}^2\tilde{\psi}_0 = 0, \quad [\vec{l}^2, \vec{\psi}] = 2\vec{\psi}, \quad [\vec{l}^2, \vec{\tilde{\psi}}] = 2\vec{\tilde{\psi}}$$

as it must be for the scalar and vector representation. The explicit solution of (11) will be given below after the parametrisation of the configuration space $(\vec{\varphi}_1, \vec{\varphi}_2)$ will be specified.

In the following we shall exploit the eigenstates of N. It is instructive to separate the fermion number operator N_F. Let: $N|\Psi\rangle = n|\Psi\rangle$ then $N_B|\Psi\rangle = (n - N_F)|\Psi\rangle$ gives

$$N_B\psi_0 = n\psi_0, \quad N_B\vec{\psi} = (n-1)\vec{\psi}, \quad N_B\tilde{\psi}_0 = (n - \frac{3}{2})\tilde{\psi}_0, \quad N_B\vec{\tilde{\psi}} = (n - \frac{1}{2})\vec{\tilde{\psi}}. \tag{12}$$

2.3 Schrödinger equation in components

The fermionic Fock space decomposition (10) for $|\Psi\rangle$ provides a matrix representation for Schrödinger equation $H|\Psi\rangle = E|\Psi\rangle$. Taking $|\Psi\rangle$ as a column one deduces H in a block-diagonal form:

$$H = \begin{pmatrix} \mathcal{H} & 0 \\ 0 & \mathcal{H}^{tr} \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} H_B & i\varphi^1 & i\varphi^2 & i\varphi^3 \\ -i\bar{\varphi}^1 & H_B & 0 & 0 \\ -i\bar{\varphi}^2 & 0 & H_B & 0 \\ -i\bar{\varphi}^3 & 0 & 0 & H_B \end{pmatrix}, \tag{13}$$

where

$$\varphi^a = \varphi_1^a + i\varphi_2^a, \quad \bar{\varphi}^a = \varphi_1^a - i\varphi_2^a. \tag{14}$$

On physical states ($\vec{j}|\Psi\rangle = 0$) the component equations for $SU(2)$ triplet $\vec{\psi}$ ($\vec{\tilde{\psi}}$) are not independent. Therefore, we are left with two sets of equations:

$$\begin{cases} H_B\psi_0 + i(\vec{\varphi}\vec{\psi}) = E\psi_0 \\ H_B\psi^3 - i\bar{\varphi}^3\psi_0 = E\psi^3 \end{cases} \tag{15}$$

and

$$\begin{cases} H_B\tilde{\psi}_0 - i(\vec{\bar{\varphi}}\vec{\tilde{\psi}}) = E\tilde{\psi}_0 \\ H_B\tilde{\psi}^3 + i\varphi^3\tilde{\psi}_0 = E\tilde{\psi}^3 \end{cases} \tag{16}$$

with $\psi^{1,2}$ ($\tilde{\psi}^{1,2}$) expressed in terms of ψ^3 ($\tilde{\psi}^3$).

3 $SU(2) \times SO(2)$ Parametrisation

3.1 Parametrisation

We parametrise the configuration space $\{\varphi_i^a\}$ by using the new coordinates $\{f, g, \theta, \gamma_1, \gamma_2, \alpha\}$ as follows :

$$\varphi_1 = \frac{1}{\sqrt{2}}U^+\left(\sigma^1 f\cos\theta - \sigma^2 g\sin\theta\right)U$$
$$\varphi_2 = \frac{1}{\sqrt{2}}U^+\left(\sigma^1 f\sin\theta + \sigma^2 g\cos\theta\right)U. \tag{17}$$

Here $SU(2)$ matrix $U(\gamma_1, \gamma_2, \alpha)$ is given by

$$U(\gamma_1, \alpha, \gamma_2) = \exp\left(\frac{i}{2}\gamma_1\sigma_3\right)\exp\left(\frac{i}{2}\alpha\sigma_1\right)\exp\left(\frac{i}{2}\gamma_2\sigma_3\right).$$

The range for the new coordinates is $0 \le \gamma_1, \gamma_2 < 2\pi$, $-\pi < \alpha < 0$, $0 \le \theta < \pi/2$, $f, g > 0$

The $SU(2)$ angular variables $\gamma_1, \alpha, \gamma_2$ are similar to the Euler angles. The $SU(2)$ adjoins φ_i can be viewed as the R^3 vectors $\vec{\varphi}_i$. Denote $\vec{e}_x, \vec{e}_y, \vec{e}_z$ the unit vectors of the fixed coordinate system. $\vec{e}_n = \vec{e}_z \times (\vec{\varphi}_1 \times \vec{\varphi}_2)$ is a vector along the knot-line (the line of intersection of the $(\vec{\varphi}_1, \vec{\varphi}_2)$ and (x, y) planes). The first rotation on γ_2, around the z-axis matches $\vec{e}_n$ with $\vec{x}$. After this rotation the vector $\vec{e}_3 = \vec{\varphi}_1 \times \vec{\varphi}_2$ falls into the (y, z) plane, with $(\vec{e}_3)_y < 0$. The second rotation around the x-axis in clockwise direction on the angle α, matches $\vec{e}_3$ with $\vec{z}$ that gives $-\pi < \alpha < 0$. These two rotations place the pair $\vec{\varphi}_1, \vec{\varphi}_2$ into the (x, y) plane and $\angle(\vec{\varphi}_1, \vec{\varphi}_2) < \pi$ ($(\vec{e}_3)_z$ is positive).

To fix the third angle γ_1, let us take a pair $(\vec{\varphi}_1, \vec{\varphi}_2)$ in (x, y) plane with $|\vec{\varphi}_1| = r_1$, $|\vec{\varphi}_2| = r_2$ and $\angle(\vec{\varphi}_1, \vec{\varphi}_2) = \beta$ and examine its orbit under the action of the rotations around the z-axis. This orbit is the set $\{(r_1\cos\delta, r_1\sin\delta), (r_2\cos(\delta+\beta), r_2\sin(\delta+\beta))\}$ with $0 \le \delta < 2\pi$. We call a pair $(\vec{\varphi}_1, \vec{\varphi}_2)$ as "$SO(2)$-orthogonal" if

$$\varphi_1^1\varphi_1^2 + \varphi_2^1\varphi_2^2 = 0, \tag{18}$$

here $\varphi_i^1 = (\vec{\varphi}_i)_x, \varphi_i^2 = (\vec{\varphi}_i)_y$. The $SO(2)$-orthogonality condition for the points of the orbit reads:

$$\tan 2\delta = -\frac{r_2^2\sin 2\beta}{r_1^2 + r_2^2\cos 2\beta}.$$

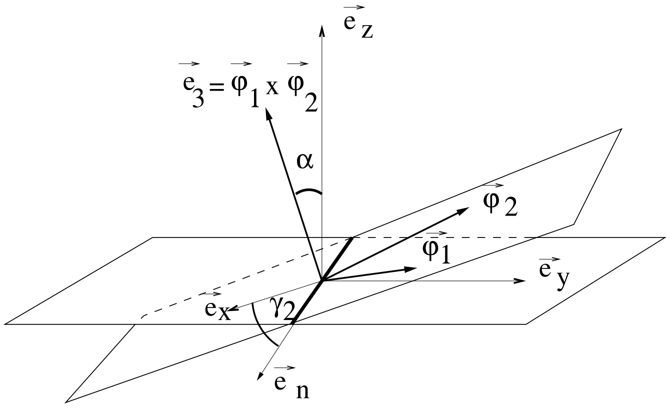

Figure 3: Angular parametrization

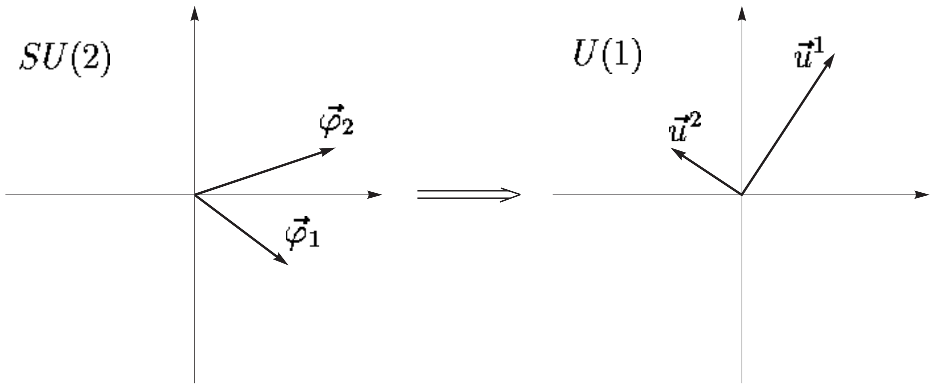

Figure 4: $SU(2)$ and $SO(2)$

Hence, at any orbit there are four $SO(2)$-orthogonal pairs $(\vec{\varphi}_1, \vec{\varphi}_2)$. One of these pairs has $\vec{\varphi}_1$ lying in the fourth quadrant *i.e.* $\varphi_1^1 > 0$, $\varphi_1^2 \leq 0$. This pair we call the special one. For any pair $(\vec{\varphi}_1, \vec{\varphi}_2)$ we define γ_1 as the angle of rotations around the $\vec{z}$-axis that matches $(\vec{\varphi}_1, \vec{\varphi}_2)$, with the special pair. The range for γ_1 is obviously $0 \leq \gamma_1 < 2\pi$.

To parametrise a special pair $(\vec{\varphi}_1, \vec{\varphi}_2)$ note that the numbers φ_i^a form also a pair of $SO(2)$ (Lorentz) orthogonal vectors: $u^1 = \{\varphi_1^1, \varphi_2^1\}$ and $u^2 = \{\varphi_1^2, \varphi_2^2\}$, so that $\arg u^2 = \arg u^1 + \frac{\pi}{2}$. The remaining coordinates $\{f, g, \theta\}$ are defined to be the polar coordinates for u^1 and u^2:

$$\begin{array}{ll} \varphi_1^1 = f\cos\theta & \varphi_1^2 = -g\sin\theta \\ \varphi_2^1 = f\sin\theta & \varphi_2^2 = g\cos\theta. \end{array} \tag{19}$$

As the pair is the special one the ranges for $\{f, g, \theta\}$ are $0 \leq \theta < \pi/2$ and $f, g > 0$.

To summarize, we have proved that by the $SU(2)$ rotation defined by the Euler-like angles γ_1, γ_2 and α any pair $(\vec{\varphi}_1, \vec{\varphi}_2)$ is matched with the special pair which is parametrized by f, g, θ. The special case of an angular momentum parametrisation was proposed in [12].

3.2 Generalized Periodicity

To extend the ranges for the angular variables γ_i up to R^1 axis we employ the usual 2π periodicity conditions for γ_i. As to the θ angle with $0 \leq \theta < \pi/2$ the situation is more subtle.

Let us take a point with $SO(2)$ angle coordinate outside the range: $(f, g, \theta + \frac{\pi}{2}, \gamma_1)$ (γ_2 and α are irrelevant for the subsequent discussion). By means of (17) this set parametrise the pair

$$\varphi_{1,2} = U^{+}(\gamma_1)\chi_{1,2}U(\gamma_1), \quad U(\gamma_1) = e^{\frac{i}{2}\gamma_1\sigma_3}$$

where

$$\begin{aligned} \chi_1 &= -\sigma_1 f \sin\theta + \sigma_2 g \cos\theta, \\ \chi_2 &= \sigma_1 f \cos\theta - \sigma_2 g \sin\theta. \end{aligned} \tag{20}$$

The pair $(\vec{\chi}_1, \vec{\chi}_2)$ is $SO(2)$-orthogonal but not special. What are the true coordinates for $\varphi_{1,2}$?

In complex notation (14) these read:

$$\varphi = \begin{pmatrix} e^{\frac{i}{2}\gamma_1} & 0 \\ 0 & e^{-\frac{i}{2}\gamma_1} \end{pmatrix}^{+} e^{i(\theta+\frac{\pi}{2})}(\sigma_1 f + i\sigma_2 g) \begin{pmatrix} e^{\frac{i}{2}\gamma_1} & 0 \\ 0 & e^{-\frac{i}{2}\gamma_1} \end{pmatrix} \tag{21}$$

$$= e^{i\theta} \begin{pmatrix} 0 & e^{-i(\gamma_1-\frac{\pi}{2})}(f+g) \\ e^{i(\gamma_1+\frac{\pi}{2})}(f-g) & 0 \end{pmatrix} = e^{i\theta} \begin{pmatrix} 0 & e^{-i(\gamma_1-\frac{\pi}{2})}(f+g) \\ e^{i(\gamma_1-\frac{\pi}{2})}(g-f) & 0 \end{pmatrix}$$

Hence, the following points must be identified

$$(f, g, \theta + \frac{\pi}{2}, \gamma_1, \gamma_2, \alpha) \cong (g, f, \theta, \gamma_1 - \frac{\pi}{2}, \gamma_2, \alpha). \tag{22}$$

Below we shall call (22) the generalized periodicity condition in θ.

It is instructive to elaborate the action of P on bosonic degrees of freedom ($\vec{\varphi}_1 \leftrightarrow \vec{\varphi}_2$). We proceed by the similar device as above. The details are listed in [1] here we quote the result:

$$\begin{aligned} \gamma_2 &\longrightarrow -\pi + \gamma_2 \\ \alpha &\longrightarrow -\pi - \alpha \\ \gamma_1 &\longrightarrow \pi - \gamma_1 \\ \theta &\longrightarrow \pi/2 - \theta \\ f &\longrightarrow f \\ g &\longrightarrow g. \end{aligned} \tag{23}$$

4 Schrödinger equation in $SU(2) \times SO(2)$ parametrisation

4.1 $N = n$ and Gauss law – an explicit solution

The $SU(2) \times SO(2)$ variables provide the explicit solution for wave function $|\Psi\rangle$ dependence on angular variables.

For N_B one deduces

$$N_B = -i\frac{\partial}{\partial\theta},$$

i.e. N_B eigenfunctions are the plane waves $e^{ik\theta}$. Applying periodicity condition (22) four times we find that wave functions components have to be 2π periodic so k must be an integer. Thus, from eqs.(12) it follows that n *can be integer or half-integer* and

$$\text{for integer } n: \quad \Psi_n \equiv \begin{pmatrix} \psi_0 \\ \vec{\psi} \\ 0 \\ \vec{0} \end{pmatrix} \qquad \text{for half-integer } n: \quad \Psi_n \equiv \begin{pmatrix} 0 \\ \vec{0} \\ \tilde{\psi}_0 \\ \vec{\tilde{\psi}} \end{pmatrix}. \tag{24}$$

In the following we shall concentrate on the case of integer n, the half-integer case will be sometimes discussed for the sake of completeness.

The components of l^a are found to coincide with the usual l^a in the Euler angles parametrisation for $SO(3)$ generators [16] up to some unsignificant modification of angles. Thus the $SU(2)$ singlet ψ_0 does not depend on $(\gamma_1, \gamma_2, \alpha)$ and has the form:

$$\psi_0 = e^{in\theta} F_0(f, g) \tag{25}$$

To write down the components of the $SU(2)$ vector $\vec{\psi}$ it is useful to slightly modify the basis by passing to the eigenvectors of l^3 (s^3):

$$\begin{aligned} |\Psi\rangle &= (\psi_0 + \psi^-(\bar{\chi}^1 + i\bar{\chi}^2)\bar{\chi}^3 + \psi^+(\bar{\chi}^1 - i\bar{\chi}^2)\bar{\chi}^3 + \psi^3\bar{\chi}^1\bar{\chi}^2 \\ &+ \tilde{\psi}_0 + \tilde{\psi}^-(\bar{\chi}^1 + i\bar{\chi}^2) + \tilde{\psi}^+(\bar{\chi}^1 - i\bar{\chi}^2) + \tilde{\psi}^3\bar{\chi}^3)|0\rangle. \end{aligned} \tag{26}$$

Then if we denote F^0, F^+, F^- the components of $\vec{\psi}$ in the "moving frame" we get by applying the $SO(3)$ rotation:

$$\psi^3 = e^{i(n-1)\theta}\left(F^0 \cos\alpha + F^+ \sin\alpha e^{i\gamma_1} + F^- \sin\alpha e^{-i\gamma_1}\right), \tag{27}$$

$$\psi^+ = \frac{1}{2} e^{i(n-1)\theta} e^{i\gamma_2}\left(F^0 \sin\alpha - F^+(\cos\alpha + 1)e^{i\gamma_1} - F^-(\cos\alpha - 1)e^{-i\gamma_1}\right), \tag{28}$$

$$\psi^- = \frac{1}{2} e^{i(n-1)\theta} e^{-i\gamma_2}\left(F^0 \sin\alpha - F^+(\cos\alpha - 1)e^{i\gamma_1} - F^-(\cos\alpha + 1)e^{-i\gamma_1}\right), \tag{29}$$

with F-s being functions of f, g and θ only.

Further restrictions and symmetry properties of F-s come from the generalized θ-periodicity (22).

Applying (22) twice we obtain restrictions dependent on the parity of n:

$$\begin{aligned} &\text{even } n \quad \Rightarrow F^0 = 0, \\ &\text{odd } n \quad \Rightarrow F_0 = F^+ = F^- = 0. \end{aligned}$$

Applying (22) once we get that only the following functions F are consistent with the generalized periodicity:

$$n = 4k : F_0(f,g) = F_0(g,f), \quad F^+(f,g) = F^+(g,f), \; F^-(f,g) = -F^-(g,f); \tag{30}$$

$$n = 4k+2 : F_0(f,g) = -F_0(g,f), \; F^+(f,g) = -F^+(g,f), \; F^-(f,g) = F^-(g,f); \tag{31}$$

$$n = 4k+1 : F^0(f,g) = F^0(g,f); \tag{32}$$

$$n = 4k+3 : F^0(f,g) = -F^0(g,f). \tag{33}$$

4.2 Hamiltonian

We deduce the explicit expression for H_B in new variables as follows. At first, we calculate the metric tensor $g^{\alpha\beta}$: $\delta^{ab}\delta_{ij} d\varphi_i^a d\varphi_j^b = g^{\alpha\beta} dx^\alpha dx^\beta$ with $\vec{x} = (f, g, \theta, \gamma_1, \gamma_2, \alpha)$. At second, we obtain the Hamiltonian as the Laplace-Beltrami operator for the metric $g^{\alpha\beta}$ plus the potential term

$$\frac{1}{8g_s} f^2 g^2.$$

This straightforward but rather tedious calculation was performed with partial use of "Maple". The result is the following:

$$\begin{aligned} H_B = -g_s \Big\{ &\Delta + D - \frac{A}{a^2}(\vec{l})^2 + \frac{1}{(b)^2}\left(c\frac{\partial^2}{\partial\theta^2} + 4a\frac{\partial}{\partial\gamma_1}\frac{\partial}{\partial\theta}\right) - \frac{b\sin 2\gamma_1}{a^2 \sin\alpha}\left(\cos\alpha\frac{\partial}{\partial\gamma_1} - \frac{\partial}{\partial\gamma_2}\right)\frac{\partial}{\partial\alpha} \\ &+ \frac{b\cos 2\gamma_1}{a^2}\frac{\partial^2}{\partial\alpha^2} - \frac{B}{a^2 b^2}\frac{\partial^2}{\partial\gamma_1^2} + \frac{b\sin 2\gamma_1}{2a^2\sin^2\alpha}\left((1+\cos^2\alpha)\frac{\partial}{\partial\gamma_1} - 2\cos\alpha\frac{\partial}{\partial\gamma_2}\right)\Big\} + \frac{1}{8g_s}a^2, \end{aligned} \tag{34}$$

here Δ is Laplacian in f, g and:

$$D = \frac{1}{J}\left(\frac{\partial J}{\partial f}\frac{\partial}{\partial f} + \frac{\partial J}{\partial g}\frac{\partial}{\partial g}\right),$$

$$a = fg, \quad b = f^2 - g^2, \quad c = f^2 + g^2, \quad J = ab,$$
$$A = f^2 \sin^2 \gamma_1 + g^2 \cos^2 \gamma_1,$$
$$B = f^6 \sin^2 \gamma_1 - 3f^4 g^2 \sin^2 \gamma_1 - 3f^2 g^4 \cos^2 \gamma_1 + g^6 \cos^2 \gamma_1.$$

Note that there are terms with the first order derivatives $\frac{\partial}{\partial f}$ and $\frac{\partial}{\partial g}$ collected in D. Although the coefficients for these terms are functions of f, g only, these terms are absent in the usual quantization of the two-dimensional toy model [17], as in the pure bosonic case as in the fermionic one[1]. These first order derivatives in couple with the zoo of angular dependent terms in (34) radically modify the potential.

4.3 Schrödinger Equation for Harmonics

Now we substitute the harmonic expansion for the wave function Ψ_n into the system (15) with the explicit Hamiltonian H_B given in (34). The result is:

for n - **odd** (only $F^0 \neq 0$) we get one PDE in f and g

$$-g_s \left\{ \Delta + D - (n-1)^2 \frac{c}{b^2} - \frac{c}{a^2} \right\} F^0 + \frac{a^2}{8g_s} F^0 = EF^0, \tag{35}$$

for n - **even** we get a system of three PDE in f and g

$$-g_s \left\{ \Delta + D - n^2 \frac{c}{b^2} \right\} F_0 + \frac{1}{8g_s} a^2 F_0 - \frac{f+g}{\sqrt{2}} F^+ + \frac{f-g}{\sqrt{2}} F^- = EF_0 \tag{36}$$

$$-g_s \left\{ \left(\Delta + D - (n-1)^2 \frac{c}{b^2} - (n-1) \frac{4a}{b^2} - \frac{(f^4+g^4)c}{2a^2 b^2} \right) F^+ - \frac{b}{2a^2} F^- \right\}$$
$$+ \frac{a^2}{8g_s} F^+ - F_0 \frac{f+g}{2\sqrt{2}} = EF^+ \tag{37}$$

$$-g_s \left\{ \left(\Delta + D - (n-1)^2 \frac{c}{b^2} + (n-1) \frac{4a}{b^2} - \frac{(f^4+g^4)c}{2a^2 b^2} \right) F^- - \frac{b}{2a^2} F^+ \right\}$$
$$+ \frac{a^2}{8g_s} F^- + F_0 \frac{f-g}{2\sqrt{2}} = EF^-. \tag{38}$$

5 The Spectrum

Remind, that for any $E \neq 0$ eigenfunctions are combined into the quartets V_{E,n_q} with quantum numbers $n_q, n_q - 1/2, -n_q + 2, -n_q + 3/2$. To examine the spectrum it is sufficient to search for any one of these four functions. One also have to take into account the constrains coming from supercharges: Q and $\bar{Q}$. The detailed analysis performed in [1] gives the significant constraint: for

$$\textbf{odd } n: \qquad Q\Psi_{En} \equiv 0. \tag{39}$$

On the other hand, it is known that $H\Psi_n = E\Psi_n$ and $Q\Psi_n = 0$ implies $\bar{Q}\Psi_n \neq 0$, i.e. for

$$\textbf{odd } n: \qquad \bar{Q}\Psi_{En} \neq 0. \tag{40}$$

We start with the case of integer n_q. (We omit the subindex q below). The content of V_{En} is the following: Ψ_n, $\Psi_{n-\frac{1}{2}}$, Ψ_{-n+2} and $\Psi_{-n+\frac{3}{2}}$ with $\bar{Q}\Psi_n = 0$ and $\bar{Q}\Psi_{-n+2} = 0$. Both n and $-n+2$ have the same parity so according to (40) odd n are forbidden. We shall examine the case of n even in Sect. 5.2.

Next we turn to a half-integer $n = n' + \frac{1}{2}$. $V_{En'+\frac{1}{2}}$ is spanned by $\Psi_{n'+\frac{1}{2}}$, $\Psi_{n'}$, $\Psi_{-n'+\frac{3}{2}}$ and $\Psi_{-n'+1}$ with $Q\Psi_{n'} = 0$, $Q\Psi_{-n'+1} = 0$. This time one of two numbers either n' or $-n'+1$ will be odd. Let, for instance, n' to be odd. As follows from (39) $Q\Psi_{n'} = 0$ and the only equation to be solved is the Schrödinger equation (35). It will be examined in the next section.

[1] The appearance of the first order derivatives upon the correct quantization was at first observed in the pioneering paper [18]

5.1 Half-integer n_q

In this section we examine the solution of eq. (35)

$$-g_s\left(\Delta + D - \frac{g^2+f^2}{g^2f^2} - (n-1)^2\frac{g^2+f^2}{(f^2-g^2)^2}\right)F^0 + \frac{f^2g^2}{8g_s}F^0 = EF^0$$

by using the Born-Oppenheimer method in the region $f \gg g$, as it is adopted in current literature. This procedure yields an approximate formulae for the spectrum and wave function. Note, that in the current variables $\|\Psi\|$ is proportional to $\int \bar{F^0}F^0 J df dg$.

To eliminate the first derivatives term D we search for the wave function in the form

$$F^0 = \frac{1}{\sqrt{J}}\varphi^0$$

that yields the equation

$$-g_s\left(\Delta + \frac{1}{4}\frac{(f^2+g^2)^3}{f^2g^2(f^2-g^2)^2} - \frac{g^2+f^2}{g^2f^2} - (n-1)^2\frac{g^2+f^2}{(f^2-g^2)^2}\right)\varphi^0 + \frac{f^2g^2}{8g_s}\varphi^0 = E\varphi^0 \tag{41}$$

and

$$\|\Psi\| = C\int \bar{\varphi}^0\varphi^0 df dg.$$

We start with introducing an extra parameter a as follows: $f \to f/a^2$ and $g \to ag$ and expanding over the powers of a. Separating the variables as $\varphi^0 = \psi_m(g|f)\chi_k(f)$ we get the leading term as

$$\left(-\frac{\partial^2}{\partial g^2} + \frac{3}{4}\frac{1}{g^2} + b^2g^2\right)\psi_m = \varepsilon_m\psi_m$$

where $b = \frac{f}{2\sqrt{2}g_s}$.

The discrete spectrum is given by $\varepsilon_m = 4b(m+1)$ where $m = 0, 1, 2, \ldots$ and ψ_m is expressed in terms of Laguerre polynomials

$$\psi_m = (-1)^m\frac{1}{\sqrt{g}}bg^2e^{-\frac{1}{2}bg^2}L^1_m(bg^2).$$

The next to leading term in the a-expansion yields the equation for χ (we omit the index k)

$$(-\frac{\partial^2}{\partial f^2} + 4\frac{f}{2\sqrt{2}g_s}(m+1))\chi = E\chi$$

where we temporary suppressed the term $1/f^2$. The change of variables $f = x + E\frac{\sqrt{2}g_s}{2(m+1)}$ yields an equation for Airy function

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\sqrt{2}(m+1)}{g_s}x\right)\chi = 0.$$

The solution in terms of $y = (\frac{\sqrt{2}(m+1)}{g_s})^{1/3}x$ is

$$\chi = \begin{cases} \sqrt{\frac{y}{3\pi}}K_{1/3}\left(\frac{2}{3}y^{3/2}\right) & \text{for} \quad y > 0 \\ \frac{\sqrt{\pi|y|}}{3}\left(J_{-1/3}\left(\frac{2}{3}|y|^{3/2}\right) + J_{1/3}\left(\frac{2}{3}|y|^{3/2}\right)\right) & \text{for} \quad y < 0. \end{cases}$$

Imposing a boundary condition $\chi(0) = 0$ we get the energy levels

$$E_{m,k} = \left(\pi\frac{3}{\sqrt{2}}\frac{(m+1)(k+\frac{3}{4})}{g_s}\right)^{2/3}$$

and

$$\varphi_{m,k} \sim c_{m,k}fg^{3/2}\xi e^{-\frac{\alpha}{2}fg^2}L^1_m(\alpha fg^2) \times \begin{cases} \frac{\sqrt{3}}{\pi}K_{1/3}\left(\frac{2}{3}\beta^{1/2}\xi^3\right), & f > \frac{1}{a}E_{m,k} \\ J_{1/3}\left(\frac{2}{3}\beta^{1/2}\xi^3\right) + J_{-1/3}\left(\frac{2}{3}\beta^{1/2}\xi^3\right), & f < \frac{1}{a}E_{m,k} \end{cases} \tag{42}$$

where

$$\alpha = \frac{1}{2\sqrt{2}g_s}, \quad \beta = \frac{\sqrt{2}(m+1)}{g_s}, \quad \xi = \left| f - \frac{1}{a}E_{m,k} \right|^{1/2}$$

The first correction to the energy is given by

$$\Delta E = \frac{\int \bar{\chi} V \chi df}{\int \bar{\chi}\chi df}.$$

In our case $V = \frac{(n-1)^2 - 1/4}{f^2}$ and the direct calculation gives

$$E = \left(\frac{3\pi}{\sqrt{2}}\right)^{2/3} \left(\frac{(m+1)(k+\frac{3}{4})}{g_s}\right)^{2/3} \left(1 + \frac{4}{3}\frac{(n-1)^2 - \frac{1}{4}}{(k+\frac{3}{4})^2}\right).$$

5.2 Even n_q and continuous spectrum

Here we apply the Born-Oppenheimer method in the region $f \gg g$ to the system (36) - (38). Our aim is to employ the toy-model-like ansatz for the wave function and justify the appearance of the continuous spectrum [2].

For $f \gg g$ the leading part in the system (36) - (38) reads

$$-(\Delta + D)\tilde{F}_0 + \frac{1}{8g_s^2}a^2\tilde{F}_0 - \frac{f}{2g_s}F^+ + \frac{f}{2g_s}F^- = E\tilde{F}_0 \tag{43}$$

$$-(\Delta + D - \frac{1}{2g^2})F^+ + \frac{a^2}{8g_s^2}F^+ + \frac{1}{2g^2}F^- - \frac{f}{2g_s}\tilde{F}_0 = EF^+ \tag{44}$$

$$-(\Delta + D - \frac{1}{2g^2})F^- + \frac{a^2}{8g_s^2}F^- + \frac{1}{2g^2}F^+ + \frac{f}{2g_s}\tilde{F}_0 = EF^- \tag{45}$$

where we rescaled $F_0 = \sqrt{2}\tilde{F}_0$.

Taking $F^- = -F^+$ we find that (44) and (45) coincide. The reduced system reads:

$$-(\Delta + D)\tilde{F}_0 + \frac{1}{8g_s^2}a^2\tilde{F}_0 - \frac{f}{g_s}\tilde{F}_0 = E\tilde{F}_0 \tag{46}$$

$$-(\Delta + D)F^+ + \frac{a^2}{8g_s^2}F^- \frac{f}{2g_s A}\tilde{F}_0 = EF^+. \tag{47}$$

It is obvious that (46) and (47) will coincide as well if one put $F^+ = \frac{1}{\sqrt{2}}\tilde{F}_0$.

In the asymptotic region the rescaling yields $D \to 1/(4g^2)$ and we are left with the equation

$$-(\Delta + \frac{1}{4g^2})\tilde{F}_0 + \frac{1}{8g_s^2}f^2g^2\tilde{F}_0 - \frac{f}{\sqrt{2}g_s}\tilde{F}_0 = E\tilde{F}_0$$

Recall, that equation

$$-(\frac{\partial}{\partial g^2} + \frac{1}{4g^2})\psi_m + \frac{1}{8g_s^2}f^2g^2\psi_m = \varepsilon_m\psi_m$$

has normalisable solutions for $\varepsilon_m = \frac{\sqrt{2}f}{g_s}(m+1/2)$, where $1/2$ is very important (not 1 !).

The next step is to solve the equation

$$-(\frac{\partial}{\partial f^2} + O(1/f^2))\chi = E\chi.$$

We confirm that for $m = 0$ the linear term from the "energy" ε_m is fully canceled by the linear potential $\frac{f}{\sqrt{2}g_s}$ thus producing the continuous spectrum.

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References

[1] I.Ya. Aref'eva, A.S. Koshelev and P. B. Medvedev *On Stable Sector in Supermembrane Matrix Model* hep-th/9911149

[2] B. de Wit, M.Lüscher and H. Nicolai, *Nucl. Phys. B* **320**(1989), 135.

[3] J. Fröhlich, J. Hoppe, *Comm. Math. Phys.* **191**(1998), 613, `hep-th/9701119`.

[4] M. Lüscher, *Nucl. Phys. B* **219**(1983), 233.

[5] B. Simon, *Ann. Phys.* **146**(1983), 209.

[6] B. V. Medvedev, *Teor. Math. Phys.* **60 N2**(1984), 224.

[7] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, *Phys. Rev. D* **55**(1997), 5112, `hep-th/9610043`.

[8] U. H. Danielsson, G. Ferretti and B. Sundborg, *Int.J.Mod.Phys. A* **11**(1996), 5463, `hep-th/9603081`.

[9] D. Kabat and P. Pouliot, *Phys. Rev. Lett.* **77**(1996), 1004, `hep-th/9603127`.

[10] M. Douglas, D. Kabat P. Pouliot and S. Shenker, *Nucl.Phys. B* **485**(1997), 85.

[11] J. Fröhlich, G.M. Graf, D. Hasler, J. Hoppe, S.-T. Yau, `hep-th/9904182`.

[12] A. M. Badalyan, *Yad. Phys.* **39**(1984), 947,
Ya. A. Simonov, *Yad.Phys.* **41**(1985), 1311.

[13] I. Ya. Aref'eva, P. B. Medvedev, O. A. Rytchkov and I. V. Volovich, `hep-th/9710032`.

[14] I. Ya. Aref'eva, A. S. Koshelev and P. B. Medvedev, *Mod.Phys.Lett. A* **13**(1998), 2481, `hep-th/9804021`.

[15] I. Aref'eva, G.Ferretti and A.Koshelev, *Mod.Phys.Lett. A* **13**(1998), 2399, `hep-th/9804018`.

[16] I. M. Gel'fand et al., *Representations of Rotation and Lorentz Group and Their Applications*, (in Russian), FizMatGiz, 1958.

[17] S.G.Matinian, G.K.Savvidy, N.G.Ter-Arutunian-Savvidi *JETP*, 53, 421(1981)
G. K. Savvidy et al., *Stochasticity of classical Yang-Mills mechanics and its elimination by higgs mechanism*, *JETP Lett.* **34**(1981), 590,
H.M. Asatrian and G.K. Savvidy, *Configuration manifold of Yang-Mills classical mechanics*, *Phys. Lett. A* **99**(1983), 290.

[18] G. K. Savvidy, *Yang-Mills quantum mechanics*, *Phys. Lett. B* **159**(1985), 325.