Solitons and Extended Objects

Eric Bergshoeff
Institute for Theoretical Physics, University of Groningen, Nijenborgh 4
9747 AG Groningen, The Netherlands

Abstract

Nonperturbative extended objects in string and/or M-theory manifest themselves as soliton-like solutions to the low-energy supergravity equations of motion. The dynamics of these objects is described by a so-called worldvolume action. Similarly, intersections of extended objects can be described by soliton-like solutions of the worldbrane equations of motion. In this talk I first review the above sketched relation between solitons and extended objects. Next, as an example, I discuss solitons on the eleven-dimensional supermembrane.

1 Extended Objects

It is a natural question to ask oneselves whether the idea of strings can be extended to higher-dimensional extended objects. In the mid-eighties it was proposed that the eleven-dimensional supergravity, put aside by string theory, is related to the existence of a supermembrane in eleven dimensions [1]. However, the quantization of such a supermembrane turned out to be problematic. In the mid-nineties the supermembrane, and other higher-dimensional extended objects, made a comeback via new developments in non-perturbative string theory. First of all, it turned out that string theory contains special nonperturbative states, the so-called BPS states, which are p-branes, i.e. extended objects with p spatial dimensions. Secondly, it was shown that the strong coupling limit of IIA supergravity, the low-energy limit of the Type IIA superstring, is given by eleven-dimensional supergravity, which is considered to be the low-energy limit of 'M-theory'. This M-theory is most likely related to a supermembrane theory in eleven dimensions.

It turns out that nonperturbative extended objects in string and/or M-theory manifest themselves as certain soliton-like solutions to the low-energy supergravity equations of motion. To get a better understanding of the nonperturbative states, it is important to study these solutions. An important class is given by the so-called "two-block" solutions. The relevant part of the Lagrangian containing only the fields carrying the solution is given by:

$$\mathcal{L}(D,a,p) = \sqrt{|g|}\left[R + \tfrac{1}{2}(\partial\phi)^2 + \frac{(-)^{p+1}}{2(p+2)!}e^{a\phi}F^2_{(p+2)}\right]\,, \tag{1}$$

where D is the dimension of the target spacetime, a the dilaton coupling parameter and p specifies the rank of the $(p+1)$-form gauge potential. This potential has a $(p+2)$-form curvature $F_{(p+2)}$. We only give here the metric of the two-block solution:

$$ds^2 = H^{-\frac{4\tilde{d}}{(D-2)\Delta}}dx^2_{(p+1)} - H^{\frac{4d}{(D-2)\Delta}}dx^2_{(D-p-1)}\,, \tag{2}$$

where $(d,\tilde{d},\Delta)$ are certain expressions of (D,a,p) and H is a harmonic function of the transverse directions:

$$H = 1 + \left(\frac{r_0}{r}\right)^{\tilde{d}}\,. \tag{3}$$

The first p+1 directions in the metric (2) are the worldvolume (WV) directions. They indicate the position of the brane in the target spacetime. The remaining $D-p-1$ directions refer to the directions transverse to the brane (see Figure 1).

As an example, relevant for this talk, I give the M2-solution in eleven dimensions:

$$\begin{aligned} ds^2 &= H^{-2/3}dx^2_{(3)} - H^{1/3}dx^2_{(8)}\,,\\ F^{(4)}_{012I} &= \partial_I H^{-1}\,, \qquad\qquad H = 1 + \frac{1}{r^6}\,. \end{aligned} \tag{4}$$

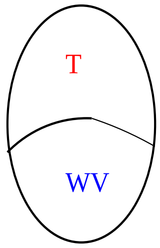

Figure 1: A two-block solution with metric (2) divides the target spacetime in $p+1$ worldvolume (WV) and $D-p-1$ transverse (T) directions.

The position of the membrane is summarized by the following figure:

$$\mathrm{M2} \quad : \quad \times | \times \times - - - - - - - - \tag{5}$$

Here the cross before the vertical bar indicates the time direction, the other two crosses indicate the position of the M2–brane worldvolume and the remaining 8 horizontal bars indicate the directions transverse to the M2–brane. Another important solution, not of the two-block kind, is the M-theory Kaluza-Klein (KK) monopole. The solution has an off-diagonal metric:

$$\begin{aligned} ds^2 &= dx^2_{(7)} - H^{-1}\left(dz - A_i dx^i\right)^2 - H dx_i^2\,, \\ F_{ij} &= \epsilon_{ijk}\partial_k H\,, \qquad\qquad H = 1 + \frac{1}{r}\,, \end{aligned} \tag{6}$$

and is summarized by the following figure:

$$\mathrm{KK} \quad : \quad \times | \times \times \times \times \times \times \; z \;\; - - - \tag{7}$$

Here z refers to a special U(1) isometry direction in the four-dimensional transverse space. Alternatively, the KK monopole metric given in (6) can be written as

$$ds^2 = ds^2(\mathbb{E}^{(1,6)}) - ds_4^2\,, \tag{8}$$

where ds_4^2 is the Taub-NUT metric

$$ds_4^2 = H^{-1}\left(dz - \mathbf{A}\right)^2 + H\, ds^2(\mathbb{E}^3)\,. \tag{9}$$

The 1-form $\mathbf{A}$ on $\mathbb{E}^3$ satisfies $\boldsymbol{\nabla} H = \boldsymbol{\nabla} \times \mathbf{A}$, which implies that H is harmonic on $\mathbb{E}^3$. The vector field $k = \partial/\partial z$ is Killing and triholomorphic. The orbits of k are Kaluza-Klein (KK) circles which shrink to points at singularities of H. Let $\mathbf{X}$ be Cartesian coordinates on $\mathbb{E}^3$ and $\mathbf{X}_0$ a constant 3-vector from the origin. The simplest choice of H that will play a role later in this talk is given by

$$H = 1 + \frac{1}{|\mathbf{X} + \mathbf{X}_0|} + \frac{1}{|\mathbf{X} - \mathbf{X}_0|}\,. \tag{10}$$

This metric describes a two-centered KK-monopole of M-theory (see Figure 2).

2 Worldvolume Solitons

Sofar, we have discussed branes as solitons with respect to the target space action. The dynamics of these branes is described by a so-called worldvolume action. In a covariant formulation the transverse position

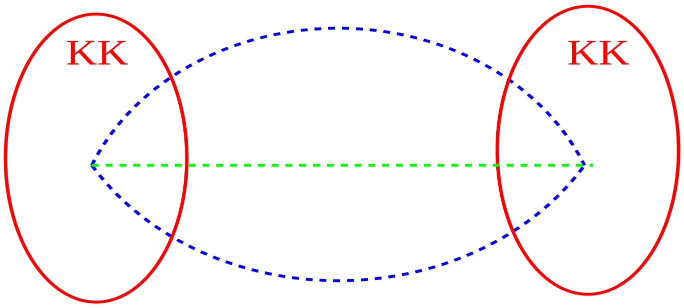

Figure 2: The figure indicates a two-centered Kaluza-Klein monopole, one positioned at $\mathbf{X} = -\mathbf{X}_0$, the other at $\mathbf{X} = +\mathbf{X}_0$. The horizontal line indicates the 3-distance between the two monopoles; the two curved lines connecting the monopoles indicate the circular z-isometry direction.

of the brane is given by embedding coordinates that are nonzero in both transverse and worldvolume directions. The ones in the worldvolume directions can be gauged away due to the reparametrization invariance on the brane worldvolume. The generic form of the worldvolume action is given by

$$\begin{aligned} S_{\mathrm{WV}} &= T\int d^{p+1}\xi e^{\alpha\phi}\sqrt{|\mathrm{det}\partial_i X^\mu \partial_j X^\nu g_{\mu\nu}|} \\ &+ Q\int d^{p+1}\xi\, C^{(p+1)} + \cdots\,, \end{aligned} \tag{11}$$

where T is the brane tension, Q the charge and $C^{(p+1)}$ the gauge field that couples to the brane. In a supersymmetric formulation there are additional fermionic fields.

Given a worldvolume action we can now search for soliton-like solutions corresponding to this action. In the same way as solitons of the target space action can be interpreted as branes moving in the target spacetime, the worldvolume solitons can be interpreted as describing intersections of two branes, see Figure 3. In other words, the common worldvolume (CW in Figure 3) behaves as a soliton moving in the worldvolume of one of the branes (CW plus one of the RT directions in Figure 3).

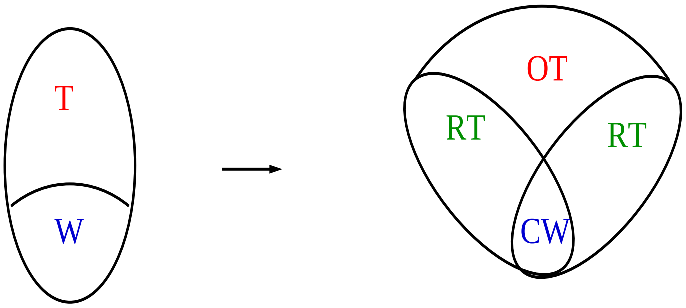

Figure 3: Two intersecting branes divide the target spacetime into four different directions: the overall transverse (OT) directions, the relative transverse (RT) directions and the common worldvolume (CW) directions.

The first example of a worldvolume soliton solution to the brane equations of motion constructed in the literature, is the Dirac Born-Infeld (DBI) soliton of [2, 3]. This solution describes the intersection of a fundamental string F1 ending on a D3–brane, see Figure 4.
The solution is carried by a nontrivial Born-Infeld (BI) worldvolume vector, describing the charged end-point of the fundamental string moving in the D3-brane worldvolume, and one active scalar describing the direction of the fundamental string. The harmonic of the solution only depends on the 3 spacelike directions in the D3-brane worldvolume that are transverse to the particle-like endpoint of the fundamental string. Other worldvolume solitons can be described in a similar way by a worldvolume gauge

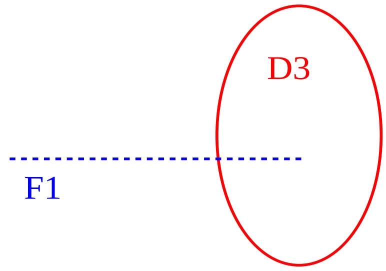

Figure 4: The DBI soliton describes a fundamental string F1 ending on a D3-brane. It corresponds to a solution of the D3-brane equations of motion with one active worldvolume vector and scalar, see the text.

field and/or a number of active scalars. For instance, the M5-brane equations of motion allow a selfdual string solution which is carried by the 2-form selfdual worldvolume potential and one active scalar [4]. It also has a 3-brane solution with two active scalars corresponding to the following intersection [4]:

$$(3|\mathrm{M5},\mathrm{M5}) \quad = \quad \left\{ \begin{array}{c|c} \times & \times\times\times\times\times - - - - - - \\ \times & \times\times\times - - \times\times - - - \end{array} \right.$$

Our goal in this talk is to extend this solitons $\leftrightarrow$ branes correspondence to sigma-model lumps in 2+1 dimensions. The idea is that any soliton solution in ordinary field theory should have a brane interpretation where the field theory is replaced by the (nonlinear) worldvolume action of a certain brane. Before specifying the brane we first summarize the ordinary sigma-model lumps in 2+1 dimensions. They are solutions of the sigma-model Lagrangean:

$$\mathcal{L} \quad = \quad \tfrac{1}{2}\int d^3x\ \eta^{ij}\partial_i X^I \partial_j X^J g_{IJ}(X) \qquad\qquad i=0,1,2\,, \tag{12}$$

where g_{IJ} is Kähler. The energy density of a solution corresponding to this sigma-model Lagrangean is given by

$$\begin{aligned} \epsilon &= \nabla X^I \cdot \nabla X^J g_{IJ} + g^{IJ} P_I P_J \\ &= \left(\nabla X^I \pm *\nabla X^K I_K{}^I\right)\left(\nabla X^J \pm *\nabla X^L I_L{}^J\right) g_{IJ} + X^{IJ}\omega_{IJ}\,, \end{aligned}$$

where $X^{IJ} \equiv \nabla X^I \times \nabla X^J$. This leads to the BPS bound

$$\epsilon \geq |X^{IJ}\omega_{IJ}|\,, \tag{13}$$

which is satisfied by

$$\nabla X^I = \mp * \nabla X^J I_J{}^I\,. \tag{14}$$

The hyper-Kähler case is similar. There is now a triplet of Kähler 2-forms $\boldsymbol{\omega_{IJ}}$ and the BPS bound is given by

$$\epsilon \geq |X^{IJ}\boldsymbol{\omega_{IJ}}|\,. \tag{15}$$

The relevant case which will play a role in the next Section is the hyper Kähler background given by a two-centered KK monopole, see Figure 2.

In the next Section we show that the sigma-model lump can be extended to a worldvolume lump of the supermembrane.

3 Solitons on the Supermembrane

Since we are considering solutions of (2+1)-dimensional sigma models it is natural to consider the eleven-dimensional membrane as the relevant brane whose worldvolume action should replace the sigma-model action (12). More specifically, following [5], we consider the supermembrane in a background for which the 4-form field-strength vanishes and the 11-metric takes the form (8).

Our starting point for finding soliton solutions as minimum energy configurations of the supermembrane will be its Hamiltonian formulation [6]. Let $\xi^i = (t, \sigma^a)$ be the worldvolume coordinates, with σ^a the worldspace coordinates, and let X^m be the D=11 spacetime coordinates. The supermembrane Lagrangian, omitting fermions, can then be written as

$$\mathcal{L} = P_m \dot{X}^m - s^a P_m \partial_a X^m - \tfrac{1}{2} v \left[P^2 + \det(g_{ab}) \right] , \tag{16}$$

where

$$g_{ab} = \partial_a X^m \partial_b X^n g_{mn} \tag{17}$$

is the induced worldspace metric, P_m is the 11-momentum conjugate to X^m, and s^a and v are Lagrange multipliers. Let $X^m = (Y^i, X^I)$ $(i = 1,2,3)$ so that

$$ds^2_{11} = dY^i dY^j \eta_{ij} + dX^I dX^J g_{IJ} , \tag{18}$$

where η is the 3-dimensional Minkowski metric. We make the gauge choice $Y^i(\xi) = \xi^i$. This implies that

$$g_{ab} = \eta_{ab} + \partial_a X^I \partial_b X^J g_{IJ} . \tag{19}$$

It also implies that

$$P_m = \left(-\varepsilon - 1, -P_I \partial_a X^I, P_I \right) , \tag{20}$$

where ε is the energy density relative to the brane vacuum (which is taken to have unit tension). The Hamiltonian constraint imposed by v can be solved for ε

$$\begin{aligned} (\varepsilon + 1)^2 \quad = \quad & 1 + \nabla X^I \cdot \nabla X^J g_{IJ} + (g^{IJ} + \nabla X^I \cdot \nabla X^J) P_I P_J \\ & + \tfrac{1}{2} (\nabla X^I \times \nabla X^J)(\nabla X^K \times \nabla X^L) g_{IK} g_{JL} , \end{aligned} \tag{21}$$

where we have used standard 2D vector calculus notation for worldspace derivatives.

We now aim to rewrite the above expression for the energy density in the form

$$\begin{aligned} (\varepsilon + 1)^2 \quad = \quad & \left(1 \pm \tfrac{1}{2} \nabla X^I \times \nabla X^J \omega_{IJ} \right)^2 \\ & + \tfrac{1}{2} \left(\nabla X^I \pm * \nabla X^K I_K{}^I \right) \left(\nabla X^J \pm * \nabla X^L I_L{}^J \right) g_{IJ} \\ & + \tfrac{1}{4} \sum_{r=1}^{R} \left(\nabla X^I \times \nabla X^J \Omega^{(r)}_{IJ} \right)^2 , \end{aligned} \tag{22}$$

where $*\nabla = (\partial_2, -\partial_1)$ if $\nabla = (\partial_1, \partial_2)$. We assume that $I_I{}^J$ is a complex structure, that the 8-metric g_{IJ} is Hermitian with respect to it and that $\omega_{IJ} = I_I{}^K g_{KJ}$ is the corresponding closed Kähler 2-form. For the moment we leave unspecified the R 2-forms $\Omega^{(r)}$. These conditions are already sufficient to ensure that all but the quartic terms in ∇X of (21) are reproduced. To reproduce the quartic terms too we require that

$$X^{IJ} X^{KL} \left[\omega_{IJ} \omega_{KL} + \sum_{r=1}^{R} \Omega^{(r)}_{IJ} \Omega^{(r)}_{KL} - 2 g_{KI} g_{JL} \right] = 0 , \tag{23}$$

where

$$X^{IJ} \equiv \nabla X^I \times \nabla X^J . \tag{24}$$

Note that X^{IJ} is an antisymmetric 8×8 matrix. If none of its 4 skew-eigenvalues vanish, then (23) implies that

$$\omega_{I(J}\omega_{K)L} + \sum_{r=1}^{R} \Omega^{(r)}_{I(J}\Omega^{(r)}_{K)L} = g_{I(K}g_{J)L} - g_{KJ}g_{IL} \,. \tag{25}$$

For a membrane in flat space this condition is satisfied by taking the matrices

$$I_I{}^J \equiv \omega_{IK}g^{KJ} \,, \qquad (J^{(r)})_I{}^J \equiv \Omega^{(r)}_{IK}g^{KJ} \tag{26}$$

to be the seven complex structures of $\mathbb{E}^8$, i.e. $R = 6$.

For every vanishing skew eigenvalue of X^{IJ} the dimension of the transverse space is effectively reduced by two. In this reduced space, we must again have (25) but it may now be possible to choose some of the $R = 6$ J matrices to vanish. For example, if X^{IJ} has two vanishing skew-eigenvalues then the transverse space is effectively 4-dimensional; in other words, there are four 'active scalars'. We may now set all but two of the J matrices to zero. The other two, together with I can be taken to be the three almost complex structures of the transverse 4-manifold (these will be covariantly constant if this transverse 4-space is hyper-Kähler, but we need not assume any special properties at this point). If X^{IJ} has three vanishing skew-eigenvalues, corresponding to two active scalars, then the transverse space is effectively two-dimensional, and we may take all the J matrices to vanish.

Given (23) we deduce that

$$\varepsilon \geq \tfrac{1}{2}|X^{IJ}\omega_{IJ}| \tag{27}$$

with equality when

$$\nabla X^I = \mp * \nabla X^J I_J{}^I \tag{28}$$

and

$$X^{IJ}\Omega^{(r)}_{IJ} = 0 \,, \qquad r = 1, \cdots, R \,. \tag{29}$$

The condition (28) is the statement that in complex coordinates Z^α, adapted to the complex structure I, the functions $Z^\alpha(z)$ are holomorphic on worldspace, with $z = \sigma^1 + i\sigma^2$. The conditions (29) are implied by (28) if the matrices $J^{(r)}$ are such that

$$IJ^{(r)} + J^{(r)}I = 0 \,, \qquad r = 1, \cdots, R \,. \tag{30}$$

This is true when $I, J^{(r)}$ are the 7 complex structures of $\mathbb{E}^8$. It is also satisfied if $I, J^{(1)}, J^{(2)}$ are the three almost complex structures of a 4-dimensional space, with the other J matrices vanishing. This is the case of most interest here because we may obviously reduce the transverse 8-space to an effective transverse 4-space by requiring all scalars to vanish except those associated with the ds_4^2 metric in (8). This restriction still allows configurations with either two or four active scalars. The situation is summarized in Table 1.

n=4	R=6	$\mathbb{E}^8$
n=2	R=2	hyper-Kahler
n=1	R=0	Kähler

Table 1: *The first column indicates the number (2n) of active scalars: the second colum gives the number R of J matrices; the third column specifies the geometry of the active 2n-dimensional transverse space*

The brane interpretation of the worldvolume lump soliton is as follows. The standard intersection of two M2-branes can be described by a lump solution of the M2-brane equations of motion with infinite energy. In this picture one of the M2-branes is the vacuum and the other M2-brane describes the intersection. The background is a flat Minkowski spacetime. To get a finite lump energy we put the two intersecting M2-branes in the background of a two-centered KK monopole (see Figure 2). This change from a flat to a KK monopole background is indicated in the figure below.

$$
n=1 \quad : \quad \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & - & - & \times & \times & - & - & - & - & - & - \end{array} \right.
$$

$$
\Downarrow
$$

$$
n=1 \quad : \quad \left\{ \begin{array}{c|cccccccccc} \times & \times & \times & - & - & - & - & - & - & - & - \\ \times & - & - & - & - & - & - & \times & \times & - & - \\ \times & \times & \times & \times & \times & \times & \times & z & - & - & - \end{array} \right.
$$

From the brane point of view, the reason that the worldvolume lump has a finite energy is due to the fact that the intersection M2-brane can wrap around the non-trivial two-cycle of the two-centered KK monopole given in Figure 2. Note that this mechanism of rendering the worldvolume lump a finite-energy solution only makes use of the fact that the Taub-Nut space of the two-centered KK monopole is a particular case of a Kähler manifold containing a nontrivial two-cycle around which the intersection M2–brane can wrap. It does not make use of the fact that the Taub-Nut space is actually hyper-Kähler.

4 Conclusions

The brane interpretation of the WV lump is consistent with the WV supersymmetry algebra. Consider the N=8, D=3 WV supersymmetry algebra:

$$
\{Q_\alpha^{\tilde{I}}, Q_\beta^{\tilde{J}}\} = \delta^{\tilde{I}\tilde{J}} P_{\alpha\beta} + Z_{(\alpha\beta)}^{(\tilde{I}\tilde{J})} + \varepsilon_{\alpha\beta} Z^{[\tilde{I}\tilde{J}]}\,, \tag{31}
$$

where the central charge $Z^{[\tilde{I}\tilde{J}]}$ is a WV 0-form and a TS 2-form. As a 8×8 matrix it has $n \leq 4$ skew-eigenvalues. The different values of n correspond to the different cases given in Table 1.

In [5] we also constructed the brane version of certain time-dependent solutions, the so-called Q-kinks. In particular, we showed that, after reduction to ten dimensions and applying T-duality, the time-dependent supermembrane Q-kink describes the migration of a (1,1)-string from one D5-brane to another D5-brane. For more details, see [5]. Recently, more work in this direction has been done in [7].

Acknowledgements

The work presented in this talk is based on the paper `hep-th/9904020` published in JHEP05 (1999) 021. I thank Paul Townsend for the stimulating collaboration on this project. This work is supported by the European Commission TMR program ERBFMRX-CT96-0045, in which I am associated to the University of Utrecht.

References

[1] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Supermembranes and eleven-dimensional supergravity*, Phys. Lett. **189B** (1987) 75; *ibidem*, *Properties of the eleven-dimensional supermembrane theory*, Ann. Phys. (N.Y.) **185** (1998) 330.
[2] C. Callan and J. Maldacena, *Brane dynamics from the Born-Infeld action*, Nucl. Phys. **B513** (1998) 198.
[3] G.W. Gibbons, *Born-Infeld particles and Dirichlet p-branes*, Nucl. Phys. **B514** (1998) 603.
[4] P.S. Howe, N.D. Lambert and P.C. West, *The Self-Dual String Soliton*, Nucl. Phys. **B515** (1998) 203; *ibidem*, *The Threebrane Soliton of the M-Fivebrane*, Phys. Lett. **B419** (1998) 79.
[5] E. Bergshoeff and P.K. Townsend, *Solitons on the supermembrane*, JHEP05 (1999) 021.
[6] E. Bergshoeff, E. Sezgin and Y. Tanii, *Hamiltonian formulation of the supermembrane*, Nucl. Phys. **B298** (1988) 187.
[7] N.D. Lambert and D. Tong, *Kinky D-strings*, `hep-th/99-7098`.