Ultraviolet Behaviour of Quantum Field Theories on Noncommutative Space-Times

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Abstract

We study properties of a scalar quantum field theory on two-dimensional noncommutative space-times. Contrary to the common belief that noncommutativity of space-time would be a key to remove the ultraviolet divergences, we show that field theories on a noncommutative plane with the most natural Heisenberg-like commutation relations among coordinates or even on a noncommutative quantum plane with $E_q(2)$-symmetry have ultraviolet divergences, while the theory on a noncommutative cylinder is ultraviolet finite. Thus, ultraviolet behaviour of a field theory on noncommutative spaces is sensitive to the topology of the space-time, namely to its compactness.

1 Introduction

The standard concept of a geometric space is based on the notion of a manifold $\mathcal{M}$ whose points $x \in \mathcal{M}$ are locally labelled by a finite number of real coordinates $x^\mu \in \mathbb{R}^4$. However, it is generally believed that the picture of space-time as a manifold $\mathcal{M}$ should break down at very short distances of the order of the Planck length $\lambda_P \approx 1.6 \times 10^{-33}\, cm$. This implies that the mathematical concepts for high energy (small distance) physics have to be changed, or more precisely, our classical geometrical concepts may not be well suited for the description of physical phenomena at small distances. No convincing alternative description of physics at very short distances is known, though different routes to progress have been proposed. One such direction is to try to formulate physics on some noncommutative space-time [1]–[5].

If the notions of the noncommutative geometry are used directly for the description of the space-time, the notion of points as elementary geometrical entity is lost and one may expect that an ultraviolet (UV) cutoff appears. The simplest model of this kind is the fuzzy sphere [4], *i.e.* the noncommutative analog of a two-dimensional sphere. As is well known from the standard quantum mechanics, a quantization of any *compact* space, in particular a sphere, leads to finite-dimensional representations of the corresponding operators, so that in this case any calculation is reduced to manipulations with finite-dimensional matrices and thus there is simply no place for UV-divergences [5]. Things are not so easy in the case of non-compact manifolds. The quantization leads to infinite-dimensional representations and we have no guarantee that noncommutativity of the space-time coordinates removes the UV-divergences. In this work we show that field theories on a noncommutative plane with the most natural Heisenberg-like commutation relations among coordinates or even on a noncommutative quantum plane with $E_q(2)$-symmetry have ultraviolet divergences, while the theory on a noncommutative cylinder is ultraviolet finite. Thus, ultraviolet behaviour of a field theory on noncommutative spaces is sensitive to the topology of the space-time, namely to its compactness.

2 Noncommutative plane with Heisenberg-like commutation relations

The noncommutative version $P^{(2)}_\lambda$ of a two-dimensional plane is obtained via replacing the commuting coordinates by the hermitian operators $\hat{x}_i$, $(i,j = 1,2)$ satisfying the commutation relations

$$[\hat{x}_i, \hat{x}_j] \; = \; i\lambda^2 \varepsilon_{ij} \; , \qquad i,j \; = \; 1,2 \; , \tag{1}$$

where λ is a positive constant of the dimension of *length*. The noncommutative analogs of field derivatives $\partial_i\hat\varphi(\hat x_i)$, $i=1,2$ are defined as

$$\partial_i\hat\varphi(\hat x_i) = \varepsilon_{ij}\frac{i}{\lambda^2}[\hat x_j,\hat\varphi(\hat x_i)]\ , \qquad i=1,2\ . \tag{2}$$

They satisfy the Leibniz rule and reduce to the usual derivatives in the commutative limit.

The free part of the field action has the form

$$S_0^{(\lambda)}[\hat\varphi,\hat\varphi^\dagger] = \mathrm{Tr}\,[(\partial_i\hat\varphi)^\dagger(\partial_i\hat\varphi)\ +\ m^2\hat\varphi^\dagger\hat\varphi]\ , \tag{3}$$

where the operator trace substitutes an integral used in the case of commutative geometry. The calculation of Green functions and other quantities in NC-QFT can be carried out in simple and natural way by the use of operator symbols. An operator symbol, as it is defined in the ordinary Quantum Mechanics, is a function on the phase space which is constructed by a definite rule from a given operator. Different rules for the construction produce different symbols (*e.g.,* Weyl, normal symbols *etc.*). Sets of such functions endowed with a so-called star-product form algebras which are isomorphic to the initial operator algebra. Knowledge of explicit forms of star-products ($\star$-products) allows to make calculations in an easy and short way.

The Weyl symbol ψ_W has some special properties which makes it especially convenient for the calculations. In particular, the explicit form of the $\star$-product which makes the algebra of Weyl symbols isomorphic to the operator algebra is defined by the expression

$$(\varphi_W\star\psi_W)\,(x) = \varphi_W(x)\exp\left\{i\frac{\lambda^2}{2}\ \overleftarrow{\partial}_i\ \varepsilon^{ij}\ \overrightarrow{\partial}_j\right\}\psi_W(x) \tag{4}$$

so that

$$\int d^2x\,(\varphi_W\star\psi_W)\,(x) = \int d^2x\varphi_W(x)\psi_W(x)\ , \tag{5}$$

because ε^{ij} is antisymmetric. Therefore, the free action of the NC-QFT in terms of the Weyl symbols has the same form as usual QFT on commutative space.

Higher order (interaction) terms contain non-locality, but analysis shows (see also [6]) that they do not remove UV-divergences. However, this is not the only possibility. Insisting only on a commutative limit condition $\lim_{\lambda\to 0}S^\lambda_{int}[\hat\varphi,\hat\varphi^\dagger] = S_{int}[\varphi,\varphi^*]$, the integrand in the noncommutative integral $I[\hat\varphi^\dagger\hat\varphi\hat\varphi^\dagger\hat\varphi]$ is defined up to the *operator ordering.* There is no problem to modify the operator ordering of the generators $\hat x_1$ and $\hat x_2$ in the integrand $\hat\varphi^\dagger\hat\varphi\hat\varphi^\dagger\hat\varphi$ in such a way that Feynman diagrams contain the exponential factors $\exp\left\{-\lambda^2k_i^2/2\right\}$. For example, one can use the normal symbols for the construction of the free action but the Weyl symbols for the interaction part. The resulting action will lead to UV-regular Feynman diagrams. However, besides this pragmatic point of view, we have not been able to find any deeper principle preferring such different ordering.

3 Noncommutative cylinder

A standard commutative cylinder with radius ρ can be identified with the set of points

$$C_\rho = \{(x,t),\ t\in\mathbb{R},\ x=\rho e^{i\phi},\ \rho=\ \mathrm{const}\,\}\ . \tag{6}$$

We shall interpret C_ρ as a space-time manifold. It is convenient to introduce $x_+=x,\ x_-=x^*$.

The starting point for development of a field theory on a non-commutative cylinder is the replacement of commutative variables $x_0,x_\pm$ by the noncommutative ones $\hat x_0,\hat x_\pm$, satisfying the $e(2)$ Lie algebra commutation relations

$$[\hat x_0,\hat x_\pm] = \pm\lambda\hat x_\pm\ , \qquad [\hat x_+,\hat x_-]=0\ . \tag{7}$$

The field $\hat\varphi=\varphi(\hat x_0,\hat x_\pm)$ is an operator in $\mathcal{L}^2(S^1,d\phi/2\pi)$ and we suppose that it possesses the expansion

$$\hat\varphi = a_0(\hat t) + \sum_{k=1}^{\infty}\left[a_k^{(+)}(\hat t)\hat x_+^k + a_k^{(-)}(\hat t)\hat x_-^k\right]\ , \tag{8}$$

The free field action has the form similar to that in the commutative case:

$$S_0[\hat{\varphi}, \hat{\varphi}^*] = \lambda \mathrm{Tr}\,\big[- \hat{\varphi}^* \big(\partial_\lambda^2 + m^2\big)\hat{\varphi}\big] \ , \tag{9}$$

where the noncommutative analog ∂_λ^2 of the operator ∂^2 is defined as

$$\partial_\lambda^2 \hat{\varphi} = -\frac{1}{\lambda^2}[\hat{x}_0, [\hat{x}_0, \hat{\varphi}]] + \frac{1}{\lambda^2 \rho^2}[\hat{x}_+, [\hat{x}_-, \hat{\varphi}]] \ . \tag{10}$$

In the noncommutative case the interaction term we take in the form $V = \frac{g}{4} : (\hat{\varphi}^\dagger \hat{\varphi})^2 :$, where the ordering $: \cdots :$ means that all expansion coefficients of fields $a_k(\hat{t}), a_k^\dagger(\hat{t})$ are collected as a right (or equivalently left) factor. The tadpole contribution is *finite*:

$$G_0(x,t;x,t) = \int_0^{\pi/\lambda} d\omega \frac{\cot\left(\sqrt{\Omega_\lambda^2(\omega) - m^2 + \mathrm{i}\varepsilon}\right)}{\sqrt{\Omega_\lambda^2(\omega) - m^2 + \mathrm{i}\varepsilon}} < \infty \ . \tag{11}$$

In the two-dimensional scalar field theory on commutative space the tadpole is the only divergent contribution to Feynman diagrams. Thus, we have shown that transition to the noncommutative cylinder leads to *UV-finite field theory.*

4 Noncommutative plane with $E_q(2)$-symmetry

In this section we consider Quantum Mechanics on a plane $P_q^{(2)}$ induced by the quantum group $E_q(2)$ [7] (see also [8] and refs therein). The elements $\bar{z},\ z$ and $\bar{\partial}_q,\ \partial_q$ defines the q-deformed algebra of functions on $P_q^{(2)}$ together with the q-deformed left-invariant vector fields (derivatives). Its defining relations read as

$$\begin{aligned}
&z\bar{z} = q^2 \bar{z} z \ , \qquad \partial_q \bar{\partial}_q = q^2 \bar{\partial}_q \partial_q \\
&\partial_q z = 1 + q^{-2} z \partial_q \ , \quad \bar{\partial}_q \bar{z} = 1 + q^2 \bar{z} \bar{\partial}_q \ , \\
&\bar{\partial}_q z = q^2 z \bar{\partial}_q \ , \qquad \partial_q \bar{z} = q^{-2} \bar{z} \partial_q \ .
\end{aligned} \tag{12}$$

In the case $q > 1$, the tadpole contribution proves to be proportional to the sum

$$\sum_{N=-\infty}^{\infty} q^{-4}(q^2-1)\Big\{[(q^N-1)^2 + (q^2-1)^2\sigma^2/q^4][(q^N+1)^2 + (q^2-1)^2\sigma^2/q^4]\Big\}^{-1/2} \ . \tag{13}$$

Since the terms in this series tend at $N \to -\infty$ to a constant which is independent of N, the series has the *linear* divergence at the lower limit (the IR-regularization parameter σ is inessential in the $N \to \pm\infty$ limit). Notice that in the nondeformed limit $q \to 1$ the series (13) coincides with the nondeformed result (logarithmic divergence). Quite similar calculation in the case $q < 1$ shows that the tadpole diagram has the linear divergence at the upper limit.

Thus the perturbation theory for the φ^4-model on the noncommutative plane $P_q^{(2)}$ with the action constructed with the help of the $E_q(2)$–quantum group invariant measure contains the UV-divergences and, hence, can not be considered as a regularization of the usual scalar field theory on the commutative plane.

5 Conclusion

We have shown that transition to a noncommutative space-time does not necessarily lead to an ultraviolet regularization of the quantum field theory constructed in this space, at least in the most natural way of introducing noncommutativity as we have performed in this paper. However, in general, theories which have the same UV-behaviour on classical spaces may acquire essentially different properties after the quantization. The reason is that quantization procedure is highly sensitive to the topology of the manifold under consideration. Thus, while in the case of classical space-time the theories on a sphere, cylinder or plane have UV-divergences, in the case of noncommutative space-time the two-dimensional theories on the fuzzy sphere and on the quantum cylinder do not have divergences at all. This can be traced to the compactness properties of the space-time in question:

It is of great interest to investigate the problem further in order to find out whether there exists any other, than the one presented in this paper, way of introducing noncommutativity so that it would remove the UV-divergences even in the case of fully noncompact space-times.

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