Dualities in Integrable Systems and N=2 SUSY Theories

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Abstract

We discuss dualities of the integrable dynamics behind the exact solution to the N=2 SUSY YM theory. It is shown that T duality in the string theory is related to the separation of variables proceedure in dynamical system. We argue that the are analogues of S duality as well as 3d mirror symmetry in the many-body systems of Hitchin type governing low-energy effective actions.

1 Integrability and low-energy effective actions for N=2 SUSY gauge theories

The description of the strong coupling regime in the quantum field theory remains a challenging problem and the main hope is connected to discovery of new proper degrees of freedom which would provide the perturbative expansion distinct from the initial one. The first sucsessful derivation of the low energy effective action in N=2 SUSY Yang-Mills theory clearly shows that solution of the theory involves new ingredients which are not familiar in this context before like Riemann surfaces and meromorphic differentials on it [1].

The general structure of the effective actions is defined by the symmetry arguments, in particular they should respect the Ward identies coming from the bare field theory. For example the chiral symmetry fixes the chiral Lagrangian in QCD and the conformal symmetry provides the dilaton effective actions in N=0 and N=1 YM theories. Since the effective actions have a symmetry origin one can expect universality properties and generically different UV theories can flow to the same IR ones. It is the symmetry origin of the effective actions that leads to the appearence of the integrable systems on the scene. The point is that the phase spaces for the integrable systems coincide with some moduli space or the cotangent bundle to the moduli space. We can mention KdV hierarchy related to the moduli of the complex structures of the Riemann surfaces, the Toda lattice related to the moduli of the flat connections or Hitchin like systems connected with the moduli of the holomorphic vector bundles. In any case moduli spaces come from some additional symmetry of the problem.

Identification of the variables in the integrable system responsible for some effective action is a complicated problem. At a moment there is no universal way to introduce the proper variables in the theories which are not topological ones but there is some experience in 2d theories [2] which suggests to identify the nonperturbative transition amplitudes among the vacuum states as the dynamical variables. As for the "space-time" variables, coupling constants and sources are the most promising candidates. It is expected that the partition function evaluated in the low-energy effective theory is the so called τ - function in integrable hierarchy which is the generating function for the conserved integrals of motion.

The particular solution of the equations of motion in the dynamical system is selected by applying the Ward identities to the partition function of the effective theory.

The arguments above explain the reason for the search of some integrable structures behind the Seiberg-Witten solution to N=2 SUSY Yang-Mills theory. This integrable structures which capture the hidden symmetry structure have been found in [3] where it was shown that A_{N_c} affine Toda chain governs the low energy effective action and BPS spectrum of pure N=2 SYM theory. The generalization to the theories with matter involves Calogero-Moser integrable system for the adjoint matter [4] and XXX spin chain for the fundamental matter [5]. In five dimensions relativistic Toda chain appears to be relevant for the pure gauge theory [6] while anizotropic XXZ chain for SQCD [8]. At the next step completely anisotropic XYZ chain has been suggested as a guide for 6d SQCD [8] while the generalization to the group product case is described by the higher spin magnets [7]. The candidate system for 6d theory with adjoint matter based on the geometry of elliptically fibered K3 manifold was suggested in [25] (see also [37]. Therefore there are no doubts in the validity of the mapping between effective low-energy efective theories and integrable finite dimensional systems.

The list of correspondences between two seemingly different issues looks as follows. The solution to the classical equation of motion in the integrable system can be expressed in terms of higher henus Riemann surface which can be mapped to the complex Liouville tori of the dynamical system. It is this Riemann surface enters Seiberg-Witten solution, and the meromorphic differential introduced to formulate the solution coincides with the action differential in the dynamical system in the separated variables. The Coulomb moduli space in N=2 theories is identified with the space of the integrals of motion in the dynamical system, for example $Tr\phi^2$ where ϕ is the adjoint scalar field coincides with the Hamiltonian for the periodical Toda system. The parameters of the field theory like masses or Λ_{QCD} determine the parameters and couplings in the integrable system. For instance in SQCD fundamental masses provide the local Casimirs in the periodical spin chains.

In spite of a lot of supporting facts it is necessary to get more transparent explanation of the origin of integrability in this context. To this aim let us discuss the moduli spaces in the problem at hands. Classically there is only Coulomb branch of the moduli space in pure gauge theory so one can expect dynamical system associated with such phase space. Coulomb branch can be considered as a special Kahler manifold [1] while the Hitchin like dynamical system responsible for the model has a hyperkahler phase space [11]. The resolution of the contradiction comes from the hidden Higgs-like branch which has purely nonperturbative nature [12, 7]. It is the dynamical system on this hidden phase space provides the integrable system of the Hitchin or spin chain type. Therefore there are two moduli spaces in our problem and one expects a pair of dynamical systems. This is what we have indeed; dynamical system on the Higgs branch yields the Hitchin like dynamics with the associated Riemann surfaces while the integrable system on the Coulomb branch gives rise to the Whitham dynamics. The "physical" meaning of the Hitchin system is to incorporate the nonperturbative instanton like contributions to the effective action in the supersymmetric way while the Whitham dynamics is nothing but the RG flows in the model [3] [1].

The next evident question is about the degrees of freedom in both dynamical systems. The claim is that all degrees of the freedom can be identified with the collective coordinates of a particular brane configuration. First let us explain where the Higgs branch comes from. The basic illustrative example for the derivation of the hyperKahler moduli space in terms of branes is the description of ADHM data as a moduli for a system of coupled D1-D5 or D0-D4 branes [15]. If the gauge fields are independent on some dimension one derives Nahm description of the monopole moduli space in terms of D1-D3 branes configuration [16]. The transition from ADHM data to the Nahm ones can be treated as a T duality transformation. At the next step the hyperkahler Hitchin space can be obtained by reducing the dependence (or additional T duality transformation) on one more dimension. This corresponds to the system of D2 branes wrapped around some surface Σ holomorphically embedded in some manifold. The most relevant example concerns T^2 embedded into K3 manifold [17]. The T duality along the torus transforms it to the system of D0 branes on the dual torus, which is the most close picture for the Toda dynamics in terms of D0 branes. The related discussion for the derivation of the Hitchin spaces in terms

[1] The latest developments within Whitham approach as well as list of references can be found in [10]

of instantons on $R^2 \times T^2$ can be found in [18].

Let us now proceed to the explicit brane picture for the N=2 theories. There are different ways to get it, one involves 10d string theory which compactified on the manifold containing the Toda chain spectral curve [20], or the M theory with M5 brane wrapped around the noncompact surface which can be obtained from the spectral curve by deleting the finite number of points [21]. This picture can be considered as the perturbative one and nonperturbative degrees of freedom have to be added. For this purpose it is useful to consider IIA projection of the M theory which involves N_c D4 branes between two NS5 branes located on a distance $\frac{l_s}{g^2}$ along, say x_6 direction. Field theory is defined on D4 branes worldvolume [22] and the extensive review cocerning the derivation of the field theories from branes can be found in [23]. The additional ingredient yielding the hidden Higgs branch comes from the set of N_c D0 branes, one per each D4 brane [12, 7]. It is known that D0 on D4 brane behaves as a abelian point-like instanton but now we have the system of interacting D0 branes. The coupling constant is provided by the Λ_{QCD} parameter which can be most naturally obtained from the mass of the adjoint scalar breaking N=4 to N=2 via dimensional transmutation proceedure.

One way to explain the need for the additional D0,s in IIA theory or KK modes in M theory looks as follows. It is known that any finite-dimensional integrable system with the spectral parameter allows the canonical transformation to the variables – spectral curve with the linear bundle. The spectral curve role is transparent and KK modes provide the linear bundle. As we have already noted they are responsible for the nonperturbative contribution but the summation of the infinite instanton sums into the finite number degrees of freedom remains the challenging problem. It is worth noting that both canonical coordinates in the dynamical system come from the coordinates of D0 branes in different dimensions. The necessity for the additional nonperturbative degrees of freedom has been also discussed in [13].

To show how the objects familiar in the integrability world translate into the brane language consider two examples. First let us consider the equations of motion in the Toda chain which has the Lax form

$$\frac{dT}{ds} = [T, A] \tag{1}$$

with some $N_c \times N_c$ matrixes T and A. The Lax matrix T can be related to Nahm matrix for the chain of monopoles using the identifications of the spectral curves for cyclic monopole configuration and periodic Toda chain [14]. All these results in the following expression for the Toda Lax operator in terms of the Nahm matrixes T_i

$$T = T_1 + iT_2 - 2iT_3\rho + (T_1 - iT_2)\rho^2 \tag{2}$$

$$\begin{aligned} T_1 &= \frac{i}{2}\sum_{j=1} q_j(E_{+j} + E_{-j}) \\ T_2 &= -\sum_{j=1} q_j(E_{+j} - E_{-j}) \\ T_3 &= \frac{i}{2}\sum_j p_j H_j, \end{aligned} \tag{3}$$

where E and H are the standard SU(N) generators, p_i, q_i represent the Toda phase space, and ρ is the coordinate on the CP^1 above. This CP^1 is involved in the twistor construction for monopoles and a point on CP^1 defines the complex structure on the monopole moduli space. With these definitions Toda equation of motion and Nahm equation acquire the simple form

$$\frac{dT}{dt} = [T, A] \tag{4}$$

with fixed A. Having in mind the brane interpretation of the Nahm data [16] we can claim that the equations of motion provide the conditions for the required supersymmetry of the whole configuration.

As another example of the validity of the brane-integrability correspondence mention the possibility to incorporate the fundamental matter in the gauge theory via branes in two ways. The first one concerns the semiinfinite D4 branes while the second one the set of N_f D6 branes. One can expect two different

integrable systems behind and they were found in [24] and [5]. It was shown in [7] that they perfectly correspond to the brane pictures and it appears that the equivalence of two representations agrees with some duality property in the dynamical system. Recently one more ingredient of the brane approach - orientifold has been recognized within intagrability approach [9]. To conclude the discussion of the many-body dynamical systems let us mention that one can inverse the logic and use the possible integrable deformations of the dynamical system to construct their field theory counterparts. Along this line of reasoning we can expect some unusual field theories with the several Λ type scales [7].

2 Dualities in integrable systems and SYM theories

We are going to study the phenomenon of *duality* whose precise definition is presented shortly. Duality is a subject of much recent investigation in the context of (supersymmetric) gauge theories, in which case the duality is an involution, which maps the observables of one theory to those of another. The duality is powerful when the coupling constant in one theory is inverse of that in another (or more generally, when small coupling is mapped to the strong one). For example, a weakly coupled (magnetic) theory can be dual to the strongly coupled (electric) theory thus making possible to understand the strong coupling behavior of the latter. In particular, it was shown [1] that using the concept of duality one can find exact low-energy Lagrangian of $\mathcal{N} = 2$,$d = 4$ $SU(2)$ gauge theory. A more fascinating recent development is that the duality connecting weak and strong coupling regimes of one or different theories may have a geometric origin. The most notorious example of that is provided by M-theory. Having in mind the relation between many-body systems and effective actions in SYM theories it is natural to obtain the natural dualities within the integrability approach. Both dynamical systems and gauge theories benefit from establishing of this correspondence which was formulated in [26, 25]. Brane picture for the Hitchin like systems presented above plays important role in derivation of the proper degrees of freedom.

2.1 T duality and separation of variables

There are three essentialy different dualities which manifest themselves in dynamical systems of the Hitchin type. Let us start with the analogue of T duality in the Hitchin like systems [26]. It appears that the proper analogue of T duality can be identified with the separation of variables in the dynamical systems.

A way of solving a problem with many degrees of freedom is to reduce it to the problem with the smaller number of degrees of freedom. The solvable models allow to reduce the original system with N degrees of freedom to N systems with 1 degree of freedom which reduce to quadratures. This approach is called a separation of variables (SoV). Recently, E. Sklyanin formulated "magic recipe" for the SoV in the large class of quantum integrable models with a Lax representation [28]. The method reduces in the classical case to the technique of separation of variables using poles of the Baker-Akhiezer function (see also [24]) for recent developments and more references). The basic strategy of this method is to look at the Lax eigen-vector (which is the Baker-Akhiezer function) $\Psi(z, \lambda)$:

$$L(z)\Psi(z, \lambda) = \lambda(z)\Psi(z, \lambda) \tag{5}$$

with some choice of normalization. The poles z_i of $\Psi(z, \lambda)$ together with the eigenvalues $\lambda_i = \lambda(z_i)$ are the separated variables. In all the examples studied so far the most naive way of normalization leads to the canonically conjugate coordinates λ_i, z_i.

Remind that the phase space for the Hitchin system can be identified with the cotangent bundle to the moduli space of holomorphic vector bundle T^*M on the surface Σ. The following symplectomorphysms can be identified with the separation of variables proceedure. The phase space above allows two more formulations; as the pair $(C, \mathcal{L})$ where C is the spectral curve of the dynamical system and $\mathcal{L}$ is the linear bundle or as the Hilbert scheme of points on $T^*\Sigma$ where the number of points follows from the rank of the gauge group. It is the last formulation provides the separated variables. The role of Hilbert schemes

on $T^*\Sigma$ in context of Hitchin system was established for the surfaces without marked points in [31] and generalized for the systems of Calogero types in [32, 33, 26].

In the brane terms separation of variables can be formulated as reduction to a system of D0 branes on some four dymensional manifold. It reminds a reduction to a system of point-like instantons on a (generically noncommutative [29]) four manifold. One more essential point is that separated variables amount to some explanation of the relation of periodic Toda chain above and monopole chains. Indeed, monopole moduli space have the structure resembling the one for the Toda chain in separated variables; both of them are the Hilbert schemes of points on the similar four manifolds.

The abovementioned constructions of the separation of variables in integrable systems on moduli spaces of holomorphic bundles with some additional structures can be described as a symplectomorphism between the moduli spaces of the bundles (more precisely, torsion free sheaves) having different Chern classes.

To be specific let us concentrate on the moduli space $\mathcal{M}_{\vec{v}}$ of stable torsion free coherent sheaves $\mathcal{E}$ on S. Let $\hat{A}_S = 1 - [\mathrm{pt}] \in H^*(S, Z)$ be the A-roof genus of S. The vector $\vec{v} = Ch(\mathcal{E})\sqrt{\hat{A}_S} = (r; \vec{w}; d - r) \in H^*(S, Z), \vec{w} \in \Gamma^{3,19}$ corresponds to the sheaves with the Chern numbers:

$$ch_0(\mathcal{E}) = r \in H^0(S; Z) \tag{6}$$

$$ch_1(\mathcal{E}) = \vec{w} \in H^2(S; Z) \tag{7}$$

$$ch_2(\mathcal{E}) = d \in H^4(S; Z) \tag{8}$$

Type IIA string theory compactified on S has BPS states, corresponding to the Dp-branes, with p even, wrapping various supersymmetric cycles in S, labelled by $\vec{v} \in H^*(S, Z)$. The actual states correspond to the cohomology classes of the moduli spaces $\mathcal{M}_{\vec{v}}$ of the configurations of branes. The latter can be identified with the moduli spaces $\mathcal{M}_{\vec{v}}$ of appropriate sheaves.

The string theory, compactified on S has moduli space of vacua, which can be identified with

$$\mathcal{M}_A = O\left(\Gamma^{4,20}\right) \backslash O(4, 20; R)/O(4; R) \times O(20; R)$$

where the arithmetic group $O(\Gamma^{4,20})$ is the group of discrete authomorphismes. It maps the states corresponding to different $\vec{v}$ to each other. The only invariant of its action is $\vec{v}^2$.

We have studied three realizations of an integrable system. The first one uses the non-abelian gauge fields on the curve Σ imbedded into symplectic surface S. Namely, the phase space of the system is the moduli space of stable pairs: $(\mathcal{E}, \phi)$, where $\mathcal{E}$ is rank r vector bundle over Σ of degree l, while ϕ is the holomorphic section of $\omega_\Sigma^1 \otimes \mathrm{End}(\mathcal{E})$. The second realization is the moduli space of pairs $(C, \mathcal{L})$, where C is the curve (divisor) in S which realizes the homology class $r[\Sigma]$ and $\mathcal{L}$ is the line bundle on C. The third realization is the Hilbert scheme of points on S of length h, where $h = \frac{1}{2}\mathrm{dim}\mathcal{M}$.

The equivalence of the first and the second realizations corresponds to the physical statement that the bound states of N $D2$-branes wrapped around Σ are represented by a single $D2$-brane which wraps a holomorphic curve C which is an N-sheeted covering of the base curve Σ. The equivalence of the second and the third descriptions is natural to attribute to T-duality.

Let us mention that the separation of variables above provides some insights on the Langlands duality which involves spectrum of the Hitchin Hamiltonians. The attempt to reformulate Langlands duality as a quantum separation of variables has been sucsessful for the Gaudin system corresponding to the spherical case [36]. The consideration in [25] suggests that the proper classical version of the Langlands correspondence is the transition to the Hilbert scheme of points on four-dimensional manifold. This viewpoint implies that quantum case can be considered as correspondence between the eigenfunctions of the Hitchin Hamiltonians and solutions to the Baxter equation in the separated variables.

2.2 S-duality

Now let us explain that S-duality well established in the field theory also has clear counterpart in the holomorphic dynamical system.

The action variables in dynamical system are the integrals of meromorphic differential λ over the A-cycles on the spectral curve. The reason for the B-cycles to be discarded is simply the fact that the B-periods of λ are not independent of the A-periods. On the other hand, one can choose as the independent periods the integrals of λ over any lagrangian subspace in $H_1(\mathbf{T}_b; Z)$.

This leads to the following structure of the action variables in the holomorphic setting. Locally over a disc in B one chooses a basis in H_1 of the fiber together with the set of A-cycles. This choice may differ over another disc. Over the intersection of these discs one has a $Sp(2m, Z)$ transformation relating the bases. Altogether they form an $Sp(2m, Z)$ bundle. It is an easy excercise on the properties of the period matrix that the two form:

$$dI^i \wedge dI^D_i \tag{9}$$

vanishes. Therefore one can always locally find a function $\mathcal{F}$ - *prepotential*, such that:

$$I^D_i = \frac{\partial \mathcal{F}}{\partial I^i} \tag{10}$$

The angle variables are uniquely reconstructed once the action variables are known.

To illustrate the meaning of the action-action AA duality we look at the two-body system, relevant for the $SU(2)$ $\mathcal{N} = 2$ supersymmetric gauge theory :

$$H = \frac{p^2}{2} + \Lambda^2 \cos(q) \tag{11}$$

with Λ^2 being a complex number - the coupling constant of a two-body problem and at the same time a dynamically generated scale of the gauge theory. The action variable is given by one of the periods of the differential pdq. Let us introduce more notations: $x = \cos(q)$, $y = \frac{p \sin(q)}{\sqrt{-2}\Lambda}$, $u = \frac{H}{\Lambda^2}$. Then the spectral curve, associated to the system which is also a level set of the Hamiltonian can be written as follows:

$$y^2 = (x - u)(x^2 - 1) \tag{12}$$

which is exactly Seiberg-Witten curve . The periods are:

$$I = \int_{-1}^{1} \sqrt{\frac{x-u}{x^2-1}} dx, I^D = \int_{1}^{u} \sqrt{\frac{x-u}{x^2-1}} dx \tag{13}$$

They obey Picard-Fuchs equation:

$$\left(\frac{d^2}{du^2} + \frac{1}{4(u^2-1)} \right) \begin{pmatrix} I \\ I^D \end{pmatrix} = 0$$

which can be used to write down an asymptotic expansion of the action variable near $u = \infty$ or $u = \pm 1$ as well as that of prepotential . The AA duality is manifested in the fact that near $u = \infty$ (which corresponds to the high energy scattering in the two-body problem and also a perturbative regime of $SU(2)$ gauge theory) the appropriate action variable is I (it experiences a monodromy $I \to -I$ as u goes around ∞), while near $u = 1$ (which corresponds to the dynamics of the two-body system near the top of the potential and to the strongly coupled $SU(2)$ gauge theory) the appropriate variable is I^D (which corresponds to a weakly coupled magnetic $U(1)$ gauge theory and is actually well defined near $u = 1$ point) . The monodromy invariant combination of the periods [27]:

$$II^D - 2\mathcal{F} = u \tag{14}$$

(whose origin is in the periods of Calabi-Yau manifolds on the one hand and in the properties of anomaly in theory on the other) can be chosen as a global coordinate on the space of integrals of motion . At $u \to \infty$ the prepotential has an expansion of the form:

$$\mathcal{F} \sim \frac{1}{2} u \log u + \ldots \sim I^2 \log I + \sum_n \frac{f_n}{n} I^{2-4n}$$

Let us emphasize that S-duality maps the dynamical system to itself. We have seen that the notion of prepotential can be introduces for any holomorphic many-body system however its physical meaning as well as its properties deserve further investigation.

2.3 "Mirror" symmetry in dynamical systems

The last type of duality we would like to discuss concerns dualities between pair of dynamical systems [34, 25]. To start with let us remind how this symmetry is formulated within the field theory. The initial motivation amounts from the 3d theory example [30] where mirror symmetry interchahges Coulomb and Higgs branches of the modili space. The specifics of three dimensions is that both Coulomb and Higgs branches are hyperkahler manifolds and the mirror symmetry can be formulated as a kind of hyperkahler rotation. The attempt to formulate the similar symmetry for 4d theory was performed in [19].

In [25] the general proceedure for the analogous symmetry within integrable system in terms of Hamiltonian and Poissonian reductions was formulated. Symmetry maps one dynamical system with coordinates x_i to another one whose coordinates coincide with the action variables of the initial system and vise versa. It appears that taking into account the relation between dynamical systems and low-energy effective actions this duality in general maps Higgs and Coulomb branches of the moduli space in the gauge theories in different dimensions.

Qualitatively this symmetry is even more transparent in terms of separated variables. As was disscussed above the proper object in separated variables is the hyperkahler four dimensional manifold which provides the phase space. In most general situation the manifold involves two tori or elliptically fibered K3 manifold. One torus provides momenta while the seconds coordinates. The duality at hands actually interchanges momentum and coordinate tori and in generic case self-duality is expected. Corresponding field theory conterpart is the hypothetical six-dimensional theory with adjoint matter.

All other cases correspond to some degeneration. Degeneration of the momentum torus to C/Z_2 corresponds to the transition to the five-dimensional theory while degeneration to R^2 corresponds to four-dimensional theory. Since the modulus of the coordinate torus has the meaning of the complixified bare coupling in the theory the interpretation of the degeneration of the coordinate torus is different. Degeneration to the cylinder corresponds to the switching off the instanton effects while the rational degeneration corresponds to the additional degeneration.

We will see below that the "mirror" symmetry maps theories in different dimensions to each other. Instanton effects in one theory "map" into the additional compact dimension in the dual counterpart. We will discuss mainly classical case with only few comments on the quantum picture. Since the wave functions in the Hitchin like systems can be identified with some solutions to the KZ or qKZ equations the quantum duality would mean some relation between solutions to the rational, trigonometric or elliptic KZ equations. Recently the proper symmetries for KZ equations where discussed in [35].

2.3.1 Two-body system (SU(2))

Let us discuss first two-body system corresponding to SU(2) case. Two-particle systems which we are going to consider reduce (after exclusion of the center of mass motion) to a one-dimensional problem. The action-angle variables can be written explicitly and the dual system emerges immediately once the natural Hamiltonians are chosen. The problem is the following. Suppose the phase space is coordinatized by (p, q). The dual Hamiltonian (in the sense of AC duality) is a function of q expressed in terms of I, φ, where I, φ are the action-angle variables of the original system : $H_D(I, \varphi) = H_D(q)$. In all the cases below there is a natural choice of $H_D(q)$.

Consider as example elliptic Calogero model whose Hamiltonian is:

$$H(p, q) = \frac{p^2}{2} + \nu^2 \wp_\tau(q). \tag{15}$$

Here p, q are complex, $\wp_\tau(q)$ is the Weierstrass function on the elliptic curve E_τ:

$$\wp_\tau(q) = \frac{1}{q^2} + \sum_{\substack{(m,n) \in Z^2 \\ (m,n) \neq (0,0)}} \frac{1}{(q + m\pi + n\tau\pi)^2} - \frac{1}{(m\pi + n\tau\pi)^2} \tag{16}$$

Let us introduce the Weierstrass notations: $x = \wp_\tau(q)$, $y = \wp_\tau(q)'$. We have an equation defining the curve E_τ:

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau) = 4\prod_{i=1}^{3}(x - e_i), \quad \sum_{i=1}^{3} e_i = 0 \tag{17}$$

The holomorphic differential dq on E_τ equals $dq = dx/y$. Introduce the variable $e_0 = 2E/\nu^2$. The action variable is one of the periods of the differential $\frac{pdq}{2\pi}$ on the curve $E = H(p,q)$:

$$I = \frac{1}{2\pi}\oint_A \sqrt{2(E - \nu^2\wp_\tau(q))} = \frac{1}{4\pi i}\oint_A \frac{dx\sqrt{x - e_0}}{\sqrt{(x-e_1)(x-e_2)(x-e_3)}} \tag{18}$$

The angle variable can be determined from the condition $dp \wedge dq = dI \wedge d\varphi$:

$$d\varphi = \frac{1}{2iT(E)}\frac{dx}{\sqrt{\prod_{i=0}^{3}(x - e_i)}} \tag{19}$$

where $T(E)$ normalizes $d\varphi$ in such a way that the A period of $d\varphi$ is equal to 2π:

$$T(E) = \frac{1}{4\pi i}\oint_A \frac{dx}{\sqrt{\prod_{i=0}^{3}(x - e_i)}} \tag{20}$$

Thus:

$$2iT(E)d\varphi = \frac{dx}{\sqrt{4\prod_{i=0}^{3}(x - e_i)}} \tag{21}$$

$$\omega d\varphi = \frac{dt}{\sqrt{4\prod_{i=1}^{3}(t - t_i)}} \tag{22}$$

where

$$\omega = -2iT(E)\sqrt{e_{01}e_{02}e_{03}} = \frac{1}{2\pi}\oint_A \frac{dt}{\sqrt{4\prod_{i=1}^{3}(t - t_i)}} \tag{23}$$

$$t = \frac{1}{x - e_0} + \frac{1}{3}\sum_{i=1}^{3}\frac{1}{e_{0i}}; t_i = \frac{1}{3}\sum_{j=1}^{3}\frac{e_{ji}}{e_{0i}e_{0j}} \tag{24}$$

where $e_{ij} = e_i - e_j$

Introduce a meromorphic function on E_τ:

$$\widehat{cn}_\tau(z) = \sqrt{\frac{x - e_1}{x - e_3}} \tag{25}$$

where z has periods 2π and $2\pi\tau$. It is an elliptic analogue of the cosine (in fact, up to a rescaling of z it coincides with the Jacobi elliptic cosine). Then we have:

$$H_D(I,\varphi) = \widehat{cn}_\tau(z) = \widehat{cn}_{\tau_E}(\varphi)\sqrt{1 - \frac{\nu^2 e_{13}}{2E - \nu^2 e_3}} \tag{26}$$

where τ_E is the modular parameter of the relevant spectral curve $v^2 = 4\prod_{i=1}^{3}(t - t_i)$:

$$\tau_E = \Big(\oint_B \frac{dt}{\sqrt{4\prod_{i=1}^{3}(t - t_i)}}\Big)\Big/\Big(\oint_A \frac{dt}{\sqrt{4\prod_{i=1}^{3}(t - t_i)}}\Big). \tag{27}$$

For large I, $2E(I) \sim I^2$.

Therefore the elliptic Calogero model with rational dependence on momentum and elliptic on coordinate maps into the "mirror" dual system with elliptic dependence on momentum and rational on coordinate. On the field theory side d=4 theory with adjoint matter maps into the d=6 theory with adjoint matter with the instanton corrections switched off. The coordinates on the Coulomb branch in d=4 theory becomes the coordinates on the "Higgs branch " in d=6 theory which explains the origin of the term "mirror" symmetry in this context.

2.3.2 Many-body systems

Now we would like to demonstrate how the "mirror" transform can be formulated in terms of Hamiltonian or Poissonian reduction proceedure. It appears that it corresponds in some sence to the simultaneous change of the gauge fixing and Hamiltonians. More clear meaning of these words will be clear from the examples below.

We summarize the systems and their duals in rational and trigonometric cases in the following table:

$$\begin{array}{ccccc} & \text{rat.CM} & \leftrightarrow & \text{rat.CM} & \\ R \to 0 & \uparrow & & \uparrow & \beta \to 0 \\ & \text{trig.CM} & \leftrightarrow & \text{rat.RS} & \\ \beta \to 0 & \uparrow & & \uparrow & R \to 0 \\ & \text{trig.RS} & \leftrightarrow & \text{trig.RS} & \end{array} \tag{28}$$

Here CM denotes *Calogero – Moser* models and RS stands for *Ruijsenaars – Schneider*. The parameters R and β here are the radius of the circle the coordinates of the particles take values in and the inverse speed of light respectively. The horizontal arrows in this table are the dualities, relating the systems on the both sides. Most of them were discussed by Simon Ruijsenaars [34]. We notice that the duality transformations form a group which in the case of self-dual systems listed here contains $\mathrm{SL}_2(Z)$. The generator S is the gorizontal arrow described below, while the T generator is in fact a certain finite time evolution of the original system (which is always a symplectomorphism, which maps the integrable system to the dual one). We begin with recalling the Hamiltonians of these systems. Throughout this section q_{ij} denotes $q_i - q_j$.

Consider the space $\mathcal{A}_{\mathbf{T}^2}$ of $SU(N)$ gauge fields A on a two-torus $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$. Let the circumferences of the circles be R and β. The space $\mathcal{A}_{\mathbf{T}^2}$ is acted on by a gauge group $\mathcal{G}$, which preserves a symplectic form

$$\Omega = \frac{k}{4\pi^2} \int \mathrm{Tr} \delta A \wedge \delta A, \tag{29}$$

with k being an arbitrary real number for now. The gauge group acts via evaluation at some point $p \in \mathbf{T}^2$ on any coadjoint orbit $\mathcal{O}$ of G, in particular, on $\mathcal{O} = \mathbb{CP}^{N-1}$. Let $(e_1 : \ldots : e_N)$ be the homogeneous coordinates on $\mathcal{O}$. Then the moment map for the action of $\mathcal{G}$ on $\mathcal{A}_{\mathbf{T}^2} \times \mathcal{O}$ is

$$kF_A + J\delta^2(p), \quad J_{ij} = i\nu(\delta_{ij} - e_i e_j^*) \tag{30}$$

F_A being the curvature two-form. Here we think of e_i as being the coordinates on $\mathbb{C}^N$ constrained so that $\sum_i |e_i|^2 = N$ and considered up to the multiplication by a common phase factor.

Let us provide a certain amount of commuting Hamiltonians. Obviously, the eigen-values of the monodromy of A along any fixed loop on $\mathbf{T}^2$ commute with themselves. We consider the reduction at the zero level of the moment map. We have at least $N-1$ functionally independent commuting functions on the reduced phase space $\mathcal{M}_\nu$.

Let us estimate the dimension of $\mathcal{M}_\nu$. If $\nu = 0$ then the moment equation forces the connection to be flat and therefore its gauge orbits are parameterized by the conjugacy classes of the monodromies around two non-contractible cycles on $\mathbf{T}^2$: A and B. Since the fundamental group $\pi_1(\mathbf{T}^2)$ of $\mathbf{T}^2$ is abelian A and B are to commute. Hence they are simultaneously diagonalizable, which makes $\mathcal{M}_0$ a $2(N-1)$ dimensional manifold. Notice that the generic point on the quotient space has a nontrivial stabilizer, isomorphic to the maximal torus T of $SU(N)$. Now, in the presence of $\mathcal{O}$ the moment equation implies that the connection A is flat outside of p and has a nontrivial monodromy around p. Thus:

$$ABA^{-1}B^{-1} = \exp(R\beta J) \tag{31}$$

(the factor $R\beta$ comes from the normalization of the delta-function). If we diagonalize A, then B is uniquely reconstructed up to the right multiplication by the elements of T. The potential degrees of freedom in J are "eaten" up by the former stabilizer T of a flat connection: if we conjugate both A and B by an element $t \in T$ then J gets conjugated. Now, it is important that $\mathcal{O}$ has dimension $2(N-1)$.

The reduction of $\mathcal{O}$ with respect to T consists of a point and does not contribute to the dimension of $\mathcal{M}_\nu$. Thereby we expect to get an integrable system. Without doing any computations we already know that we get a pair of dual systems. Indeed, we may choose as the set of coordinates the eigen-values of A or the eigen-values of B.

The two-dimensional picture has the advantage that the geometry of the problem suggest the $SL_2(Z)$-like duality. Consider the operations S and T realized as:

$$S : (A, B) \mapsto (ABA^{-1}, A^{-1}); \quad T : (A, B) \mapsto (A, BA) \tag{32}$$

which correspond to the freedom of choice of generators in the fundamental group of a two-torus. Notice that both S and T preserve the commutator $ABA^{-1}B^{-1}$ and commute with the action of the gauge group. The group Γ generated by S and T in the limit $\beta, R \to 0$ contracts to $\mathrm{SL}_2(Z)$ in a sense that we get the transformations by expanding

$$A = 1 + \beta P + \ldots, \quad B = 1 + RQ + \ldots$$

for $R, \beta \to 0$.

The disadvantage of the two-dimensional picture us the necessity to keep too many redundant degrees of freedom. The first of the contractions actually allows to replace the space of two dimensional gauge fields by the cotangent space to the (central extension of) loop group:

$$T^*\hat{G} = \{(g(x), k\partial_x + P(x))\}$$

which is a "deformation" of the phase space of the previous example ($Q(x)$ got promoted to a group-valued field). The relation to the two dimensional construction is the following. Choose a non-contractible circle $\mathbf{S^1}$ on the two-torus which does not pass through the marked point p. Let x, y be the coordinates on the torus and $y = 0$ is the equation of the $\mathbf{S^1}$. The periodicity of x is β and that of y is R. Then

$$P(x) = A_x(x, 0), g(x) = P \exp \int_0^R A_y(x, y) dy.$$

The moment map equation looks as follows:

$$kg^{-1}\partial_x g + g^{-1}Pg - P = J\delta(x), \tag{33}$$

with $k = \frac{1}{R\beta}$. The solution of this equation in the gauge $P = \mathrm{diag}(q_1, \ldots, q_N)$ leads to the Lax operator $A = g(0)$ with R, β exchanged. On the other hand, if we diagonalize $g(x)$:

$$g(x) = \mathrm{diag}\left(z_1 = e^{iRq_1}, \ldots, z_N = e^{iRq_N}\right) \tag{34}$$

then a similar calculation leads to the Lax operator

$$B = P \exp \oint \frac{1}{k} P(x) dx = \mathrm{diag}(e^{i\theta_i}) \exp iR\beta\nu \mathrm{r}$$

with

$$\mathrm{r}_{ij} = \frac{1}{1 - e^{iRq_{ji}}}, i \neq j; \quad \mathrm{r}_{ii} = -\sum_{j \neq i} \mathrm{r}_{ij}$$

thereby establishing the duality $A \leftrightarrow B$ explicitly.

When Yang-Mills theory is formulated on a cylinder with the insertion of an appropriate time-like Wilson line, it is equivalent to the Sutherland model describing a collection of N particles on a circle. The observables $\mathrm{Tr}\phi^k$ are precisely the integrals of motion of this system. One can look at other supercharges as well. In particular, when the theory is formulated on a cylinder there is another class of observables annihilated by a supercharge. One can arrange the combination of supercharges which will annihilate the Wilson loop operator. By repeating the procedure similar to the one in one arrives at the quantum mechanical theory whose Hamiltonians are generated by the spatial Wilson loops. This model is nothing but the rational Ruijsenaars-Schneider many-body system.

The self-duality of trigonometric Ruijsenaars system has even more transparent physical meaning. Namely, the field theory whose quantum mechanical avatar is the Ruijsenaars system is three dimensional Chern-Simons theory on $\mathbf{T}^2 \times \mathbf{R}^1$ with the insertion of an appropriate temporal Wilson line and spatial Wilson loop. It is the freedom to place the latter which leads to several equivalent theories. The group of (self-)dualities of this model is very big and is generated by the transformations S and T .

Finally let us comment on six dimensional theory compactified on a three dimensional torus $\mathbf{T}^3$ down to three dimensions. As was discussed extensively in [38] in case where two out of three radii of $\mathbf{T}^3$ are much smaller then the third one $\mathbf{R}$ the effective three dimensional theory is a sigma model with the target space $\mathcal{X}$ being the hyper-kahler manifold (in particular, holomorphic symplectic) which is a total space of algebraic integrable system. The complex structure in which $\mathcal{X}$ is the algebraic integrable system is independent of the radius $\mathbf{R}$ while the Kähler structure depends on $\mathbf{R}$ in such a way that the Kähler class of the abelian fiber is proportional to $1/\mathbf{R}$.

The theory in three dimensions which came from four dimensions upon a compactification on a circle whose low-energy effective action describes only abelian degrees of freedom can be always dualized to the theory of scalars/spinors only, due to the vector-scalar duality in three dimensions. In this way different sets of vector and hypermultiplets in four dimensions can lead to the same three dimensional theory [30, 19].

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