Self-action on locally flat spaces with conical singularities

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Abstract

Potential of the field of a point source on the multicentre $3D$ spacetime is determined. General expressions for the regularized self-energy, self-force and moment of force are found. It is shown that self-action effects near the cone and relative contributions of higher multipoles increase infinitely when angular deficit tends to 2π. These results are generalized for the case of parallel strings one of which is a currentcarrying one. The problems of mass and tension renormalization are also considered.

1. Conical spaces appear in consideration of physical phenomena at very different space-time scales. Cosmic string [1] is the most popular but not the only physical system which generates conical singularities in a spacetime. Other examples can be found in the physics of black holes [2], cosmology [3], solid state physics [4]. In the simplest cases conical space is a direct product of time or Minkowski plain and a locally flat surface with N conical singularities. In the first case one gets $3D$ ultrastatic spacetime of N point masses (the so called multicentre spacetime) [5], in the second case we obtain $4D$ ultrastatic spacetime of N parallel cosmic strings [6]. Riemann tensor of these spaces vanishes everywhere except on the tops of the cones, where it has δ-like singularities [7]. So, conical defects change the global properties of the spacetime only, and the effects of conical structure on classical or quantized matter fields are purely nonlocal (topological).

In the last years the interest in cosmic strings has been associated with investigations in the quantum field theory. The vacuum energy has been calculated for the massive and massless fields of different spins at zero [8] and finite [9] temperatures. It was shown [10] that the additional deformation of the vacuum loops due to the multiconical boundary conditions leads to the Casimir-like interaction between the strings, and the strings of the multiple cosmic string spacetime which have no purely gravitational interaction are attracted to each other with a force that decreases as the third power of the distance between them.

The effect of a conical defect on quantized fields is purely topological. From the formal point of view it is because of the fact that the solutions of a partial differential equation are sensitive to the global structure of a spacetime. Therefore, similar effects must take place in the classical field theory too. The repulsion of a charged particle [11] and the attraction of a gravitating one [12] are the well known examples of this type of effects.

In the papers mentioned previously the simplest sources without any internal structure were considered. However, from the low-dimensional gravity we know that if the charge distribution has a finite size and cannot be described by the total charge only, the problem of the topological contribution to an interaction energy arises. It was shown [13] that this contribution is of the same order of magnitude as the topological contribution to the self-energy and does not tend to zero in the limit of infinitely small source if its multipole structure is fixed.

Moreover, the effects were studied in the spacetime of a single conical defect. In typical cosmological scenarios, however, strings are generated in the form of random network of straight moving segments, and there are about ten straight strings under the Hubble horizon at our era [14]. Another motivation comes from the solid state physics because the number of disclinations per unit volume in a real crystal exceeds the unity by far. Thus, in some cases multiconical spacetime is much more appropriate model than the particular case of only one conical defect. But the work on such spaces is much more complicated. Indeed, the spacetime of a single defect possesses enough Killing vectors to separate the variables in a field equation. The loss of the azimuthal symmetry, as it is in the multiconical case, makes impossible to proceed along this line. One possible approach to this problem is associated with the use of perturbation theory [10], another one is based on the possibility to solve some problems explicitly in low-dimensional gravity [13].

In this paper we present an explicitly solvable problem on the $3D$ multicentre and $4D$ multistring spacetimes. Our paper completes the investigations that were initiated in the papers [13, 15, 16].

2. Let us consider electric field of a static charge distribution $j(x)$ in the $3D$ spacetime with the metric

$$ds^2 = -dt^2 + \gamma_{ab}(x^c)dx^a dx^b \qquad a,b,c = 1,2\ . \tag{1}$$

To solve the problem we should find the Green function of the Poisson equation

$$\Delta_{x''} G(x', x'') = \delta(x', x'')\ . \tag{2}$$

As it was shown by one of the authors [13], the problem of finding $G(x', x'')$ is greatly simplified by the fact that any two-dimensional Riemannian surface is locally conformal to the Euclidean plane. This means that a coordinate system can be introduced, in which in the vicinity of any point on the two-dimensional section $t = const$ the metric takes the form

$$\gamma_{ab}(x) = e^{-\Lambda(x)}\delta_{ab}\ . \tag{3}$$

Indeed, in conformal coordinates the two-dimensional Poisson equation (2) reduces to the one which has exactly the same form as the Poisson equation on the Euclidean plane. Thus, in the case of an infinite $2D$ Riemann surface which is covered by the conformal coordinates globally this Green function amounts to

$$G(x', x'') = \frac{1}{2\pi} \ln|x' - x''|\ , \tag{4}$$

where $|x' - x''|$ is the Euclidean norm of the conformal vector (x'-x") (hear x^a are the conformal coordinates on the section $t = const$).

We are interested in the behavior of the Green function and its derivatives in the coincidence limit $x', x'' \to x$. In this limit all these quantities diverge. To obtain a regularized value of the Green function, it is necessary to subtract from (4) the Green function on the Euclidean plane

$$G^{reg}(x', x'') = G(x', x'') - G_E(x', x'')\ , \quad G_E(x', x'') = \frac{1}{4\pi} \ln 2\sigma(x', x'')\ , \tag{5}$$

where $\sigma(x', x'')$ is a half of the squared geodesic distance between the points x' and x''. Let x be the central point of the geodesic. Then using the well-known technique [17] we obtain

$$G^{reg}(x', x'') = \frac{\Lambda(x)}{4\pi} + \frac{\sigma^a\sigma^b}{96\pi}\Big(\nabla_a\nabla_b\Lambda + \nabla_a\Lambda\nabla_b\Lambda - \frac{\gamma_{ab}}{2}\nabla_c\Lambda\nabla^c\Lambda\Big) + O(\sigma^2)\ , \tag{6}$$

where $\sigma^a \equiv \epsilon^a_{a''}\sigma^{a''}(x', x'')$ is a tangent vector to the geodesic at the point x directed to the point x'' and $\epsilon^a_{a''}$ stands for the bivector of parallel transport from x'' to x.

In the last expression Λ is the function of x, and therefore it depends on x' and x''. Taking it into account we can obtain corresponding expansion for the first covariant derivative of the Green function

$$\nabla_{a'} G^{reg}(x', x'') = \epsilon^a_{a'}\frac{\nabla_a\Lambda}{8\pi} - \frac{\epsilon^a_{a'}\sigma^b}{48\pi}\Big(\nabla_a\nabla_b\Lambda + \nabla_a\Lambda\nabla_b\Lambda - \frac{1}{2}\gamma_{ab}\nabla_c\Lambda\nabla^c\Lambda\Big) + O(\sigma)\ . \tag{7}$$

From (6) and (7) we obtain that in the coincidence limit

$$[G^{reg}] = \frac{\Lambda}{4\pi} \quad , \qquad [G^{reg}_{a'}] = \frac{\nabla_a\Lambda}{8\pi}\ . \tag{8}$$

Here and below we use Synge's bracket notation [18]

$$[G^{reg}_{a'_1 \dots a'_n b''_1 \dots b''_m}] = \nabla_{a'_n} \dots \nabla_{a'_1} \nabla_{b''_m} \dots \nabla_{b''_1} G^{reg}(x', x'')\Big|_{x'=x=x''}\ . \tag{9}$$

It can be shown that the second derivative of the Green function is equal to

$$[G^{reg}_{b''a'}] = \frac{1}{48\pi}\Big(2\nabla_a\nabla_b\Lambda - \nabla_a\Lambda\nabla_b\Lambda + \frac{1}{2}\gamma_{ab}\nabla_c\Lambda\nabla^c\Lambda\Big)\ . \tag{10}$$

The expressions for higher derivatives are too long to be written here. Some of them one can find in our paper [19].

3. We are now in a position to consider a point source localized at P which obeys some nontrivial multipole structure.

The most convenient way to define corresponding multipole moments is the use of the Riemannian coordinates. Choosing the origin of the Riemannian frame y at P, we obtain

$$d^2s = -d^2t + \left(\delta_{ab} - \frac{1}{3}R_{acbd}(P)y^c y^d + O(y^3)\right)dy^a dy^b \ . \tag{11}$$

Suppose that when the support of $j(y)$ collapses to the point P all the limits

$$\lim_{suppj\to\{0\}} \int d^2y\sqrt{\gamma(y)}y^{a_1}y^{a_2}...y^{a_n}j(y) = D^{a_1a_2\dots a_n} \tag{12}$$

are well defined and some of them are not equal to zero.

In our case $D^{a_1a_2\dots a_n}$ are the components of the symmetric tensor $\mathbf{D^{(n)}} \in T_P \times T_P \times \dots \times T_P \quad n\ times$ (T_P is the tangent plane to the surface $t = const$ at P) which determines corresponding multipole moment.

Formal expression for the total energy of the field can be written as follows

$$\mathcal{E} = -2\pi \int d^2x' d^2x'' \sqrt{\gamma(x')}\sqrt{\gamma(x'')}j(x')G(x',x'')j(x'') \ . \tag{13}$$

Here G stands for the Green function of the two-dimensional equation (2).

Regularizing $\mathcal{E}$ according to the prescription (5), we obtain that in the limit $suppj(x) \to \{0\}$ (for details see [19])

$$\mathcal{E}^{reg} = -2\pi \sum_{n,m} \frac{1}{n!m!}\mathbf{D}^{(n)}\left[\mathbf{G}^{\mathbf{reg}}_{(\mathbf{n}'\mathbf{m}'')}\right]\mathbf{D}^{(m)} \ , \tag{14}$$

where we introduce the notation

$$\mathbf{D}^{(n)}\left[\mathbf{G}^{\mathbf{reg}}_{(\mathbf{n}'\mathbf{m}'')}\right]\mathbf{D}^{(m)} = D^{a_1\dots a_n}D^{b_1\dots b_m}\left[G^{reg}_{a'_1\dots a'_n b''_1\dots b''_m}\right] \ . \tag{15}$$

We see that $\mathcal{E}^{reg}$ depends on coordinates and orientation of the multipole moments. So, we can expect the existence of the both topological force and moment of force.

Proceeding along the same line as in the case of self-energy considered above, one can derive the expression for the self-force

$$\mathcal{F}^{reg}_a = 2\pi \sum_{nm} \frac{1}{n!m!}\left(\mathbf{D}^{(n)}\left[\nabla_{a'}\mathbf{G}^{\mathbf{reg}}_{(\mathbf{n}'\mathbf{m}'')}\right]\mathbf{D}^{(m)} + \mathbf{D}^{(n)}\left[\nabla_{a''}\mathbf{G}^{\mathbf{reg}}_{(\mathbf{n}'\mathbf{m}'')}\right]\mathbf{D}^{(m)}\right) \ . \tag{16}$$

We see that after the use the Synge's theorem [18]

$$\left[G_{a'_1\dots a'_n b''_1\dots b''_m c'}\right] + \left[G_{a'_1\dots a'_n b''_1\dots b''_m c''}\right] = \left[G_{a'_1\dots a'_n b''_1\dots b''_m}\right]_{;c} , \tag{17}$$

the force can be written as follows

$$\mathcal{F}^a = -\gamma^{ab}\nabla_b\mathcal{E}^{reg} \ . \tag{18}$$

Corresponding expression for the self-moment reads [19]

$$\mathcal{K}^{reg} = 2\pi \sum_{nm} \frac{1}{n!m!}\left(\partial_\phi\mathbf{D}^{(n)}\left[\mathbf{G}^{\mathbf{reg}}_{(\mathbf{n}'\mathbf{m}'')}\right]\mathbf{D}^{(m)} + \mathbf{D}^{(n)}\left[\mathbf{G}^{\mathbf{reg}}_{(\mathbf{n}'\mathbf{m}'')}\right]\partial_\phi\mathbf{D}^{(m)}\right) , \tag{19}$$

where

$$\partial_\phi D^{a_1\dots a_n} = \sum_{k=1}^{n} D^{a_1\dots a_{k-1}\ b\ a_{k+1}\dots a_n}E_{bc}\gamma^{ca_k} \ , \tag{20}$$

and $E^{ab} = \varepsilon^{ab}/\sqrt{\gamma}$.

Thus, we obtain

$$\mathcal{K} = -\frac{\partial}{\partial\phi}\mathcal{E}^{reg} \ . \tag{21}$$

General expressions for the regularized self-energy, force and moment of force will be used in the next sections.

4. Our results enable one to consider the case of a locally flat $3D$ spacetime of N point masses. As it was shown in [5], in this case the metric has the form (1), (3) with the conformal factor

$$\Lambda(x) = \sum_{i=1}^{N} 2(1-b_i)\ln r_i \ , \tag{22}$$

where r_i is a conformal distance from the ith gravitating particle and b_i determines the angle deficit of the corresponding conical singularity ,$\Delta\phi_i = 2\pi(1-b_i)$.

In the previous sections we show that all the calculations are straightforward. So, let us consider in more detail the particular case of only one conical singularity which is very interesting from the point of view of physical interpretation. To make this interpretation more obvious, let us introduce a new radial coordinate ρ which gives the geodesic distance from the cone [5]

$$b\rho = r^b \ . \tag{23}$$

Substituting (22) into Eq. (14) and taking into account (23) we obtain that the topological self-energy of the point multipole near the cone is the same as the interaction energy between two similar multipoles in the $3D$ Minkowski spacetime. Indeed, a straightforward calculation leads to the result that from the point of view of the inertial observer which is located at the source, the topological self-energy of a simple charge $q = D^{(0)}$ takes the form

$$\mathcal{E}^{reg} = -2qq_{ind}\ln b\rho \quad with \quad q_{ind} = q(1-b)/2b \ . \tag{24}$$

Self-energy of a dipole moment $\mathbf{d} = \mathbf{D}^{(1)}$ coincides with the interaction energy of two (anti)parallel dipoles

$$\mathcal{E}^{reg} = -(\mathbf{d}\mathbf{E}_{ind}) \ , \quad \mathbf{E}_{ind} = 2\frac{2\boldsymbol{\rho}(\boldsymbol{\rho}\mathbf{d}_{ind}) - \rho^2\mathbf{d}_{ind}}{\rho^4} \ , \tag{25}$$

where the induced dipole moment is equal to

$$\mathbf{d}_{ind} = -\frac{(1-b^2)}{24b^2}\mathbf{d} \ . \tag{26}$$

For the induced quadrupole moment one gets

$$D^{ab}_{ind} = \frac{D^{ab}}{1440b^4}\left(11b^2+1\right)\left(1-b^2\right) \ . \tag{27}$$

From (24), (26) and (27) we see that topological self-energy tends to infinity when b tends to zero. Moreover, the relative contribution of higher multipoles increases too.

Some words about the mass renormalization. For the sake of simplicity let us consider a simple point charge. In this case the divergent part of the action reads

$$S^{div} = 2\pi q^2 G_E(x,x)\int dt \ . \tag{28}$$

Introducing an ultraviolet cutoff δ

$$G_E(x',x''|\delta) = -\frac{1}{4\pi}\ln\Big(2\sigma(x',x'')+\delta^2\Big) \qquad \delta\to 0 \ , \tag{29}$$

we can rewrite S^{div} in the form

$$S^{div}(\delta) = q^2\ln\frac{\Lambda}{\delta}\int dt \ . \tag{30}$$

In this expression Δ denotes an arbitrary renormalization length which must be introduced because of the logarithmic nature of the divergence. It was emphasized by Dirac [20], that the infinite self-action problem is cured by renormalizing the mass. In the case under consideration the ultraviolet divergent bare mass of the charge reads

$$m(\delta) = m_{ren} + q^2 \ln \frac{\Lambda}{\delta} \ , \tag{31}$$

with m_{ren} being the renormalized mass of the charge.

5. Now let us consider background $4D$ space-time with the metric

$$ds^2 = -dt^2 + dz^2 + \gamma_{ab}(x^c)dx^a dx^b \qquad a, b, c = 1, 2 \ . \tag{32}$$

On this background we shall consider a stationary distribution of a four-current

$$j^\mu(x) = (j^t(x), 0, 0, j^z(x)) \ , \tag{33}$$

which is supposed to be time independent and invariant under translations along the z axis. In this case the problem becomes effectively two-dimensional and the expression for self-energy consists of two components which depend on j^t and j^z correspondingly.

In the case of infinitely thin current (currentcarrying string) the first term looks like the self-energy in three dimensions (14), while the second one is of the same form, but every electric moment $\mathbf{D^{(n)}}$ must be replaced by the magnetic one $\mathbf{M^{(n)}}$

$$\mathcal{E}_l^{reg} = \frac{\mathcal{E}^{reg}}{\int dz} = -2\pi \sum_{n,m} \frac{1}{n!m!} \left(\mathbf{D}^{(n)} \left[\mathbf{G^{reg}_{(n'm'')}} \right] \mathbf{D}^{(m)} + \mathbf{M}^{(n)} \left[\mathbf{G^{reg}_{(n'm'')}} \right] \mathbf{M}^{(m)} \right) \ . \tag{34}$$

Corresponding expressions for the self-force and for the moment of self-force read [19]

$$\mathcal{F}_l^a = -\gamma^{ab} \nabla^b \left(\mathcal{E}_{l\ E}^{reg} - \mathcal{E}_{l\ H}^{reg} \right) \quad and \quad \mathcal{K}_l = -\frac{\partial}{\partial \phi} \left(\mathcal{E}_{l\ E}^{reg} - \mathcal{E}_{l\ H}^{reg} \right) \ . \tag{35}$$

Now we can take Λ in the form (22) and consider the spacetime of multiple cosmic strings [6]. Some nontrivial features of the topological self-action on this spacetime were considered in the papers [15, 16]. Here we should like to add some comments about the problem of tension renormalization.

This problem arises for self-interacting infinitely thin cosmic strings and its consideration can be performed along the same line as in the three-dimensional case. But the result can be predicted on the purely qualitative level in close connection with the symmetries of the system under consideration.

First of all, let us remember that $z\partial/\partial t + t\partial/\partial z$ is one of the Killing vectors of the spacetime of multiple cosmic strings (the spacetime of N parallel strings). Indeed, in the case of a straight string such a Lorentz transform changes the parametrization of the string's world sheet, but the Nambu action is invariant under general reparametrizations of the world sheet. The existence of this symmetry means that the longitudinal motion of a Nambu string is unobservable.

To obtain a time independent infinitely thin current we can use any action for the superconducting string, say the Nielsen-Olesen one [21]. But the action for the superconducting string is invariant under the reparametrization of the string's world sheet too. Thus, if we consider N parallel strings, and one of the strings is a superconducting string, a Lorentz boost along them leads to the reparametrization of the sheets, and so must be unobservable. This means that renormalizing string's tension counterterm must depend on $j^2 = (j^t)^2 - (j^z)^2$, and the only possibility to obtain the expression for the renormalized tension which is in agreement with (31) is to write

$$\mu(\delta) = \mu_{ren} + j^2 \ln \frac{\Lambda}{\delta} \ . \tag{36}$$

Thus, for the isotropic current the divergent part of the action (and self-force) is shown to be exactly zero.

6. In this paper we consider some explicitly solvable problems which enable one to establish new details in the topological interaction of conical singularities with a physical system. The primary goal of our paper was to show that the effect of topological self-action which is induced by the global conical structure of a spacetime is much more nontrivial and much less studied phenomenon than it seems from the first glance. We show that conical singularity has a lot of faces, and its interaction with a physical system depends not on integral properties of the system only, but on its internal structure, and more complicated is the system, more nontrivial will be its interaction with a (multi)cone.

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