Zero Locus Reduction of the BRST Differential

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Abstract

I point out an unexpected relation between the BV (Batalin–Vilkovisky) and the BFV (Batalin–Fradkin–Vilkovisky) formulations of the same pure gauge (topological) theory. The *nonminimal sector* in the BV formulation of the topological theory allows one to construct the Poisson bracket and the BRST charge on some Lagrangian submanifold of the BV configuration space; this Lagrangian submanifold can be identified with the phase space of the BFV formulation of the same theory in the *minimal* sector of ghost variables. The BFV Poisson bracket is induced by a natural even Poisson bracket on the stationary surface of the master action, while the BRST charge originates from the BV gauge-fixed BRST transformation defined on a gauge-fixing surface. The inverse construction allows one to arrive at the BV formulation of the topological theory starting with the BFV formulation. This correspondence gives an intriguing geometrical interpretation of the nonminimal variables and clarifies the relation between the Hamiltonian and Lagrangian quantization of gauge theories.

1 Introduction

The aim of this talk is to outline an unexpected relation between the Batalin–Vilkovisky [1] quantization and its Hamiltonian counterpart known as the the Batalin–Fradkin–Vilkovisky quantization [2]. I investigate the pure gauge sector of a gauge theory. The relevant models are the topological theories, i.e. the theories all of whose degrees of freedom are gauge ones. To anticipate the result, the correspondence is based on the zero locus reduction [3, 4], i.e., the construction of an even (odd) Poisson bracket on the zero locus of the odd nilpotent vector field on the (anti)symplectic manifold.

A geometrical counterpart of the BV quantization configuration space (see [12]–[16] for the geometrically covariant formulation of the BV formalism) is the QP manifold (the manifold equipped with an antibracket and a compatible odd nilpotent vector field Q; in the BV context, Q is usually given by $\mathrm{Q} = (S, \cdot)$, where S is the BV master action satisfying the master equation $(S, S) = 0$). As shown in [3, 4], the zero locus of the odd vector field Q on a QP manifold is an even symplectic manifold, provided the appropriate nondegeneracy condition is imposed on Q. In the BV context, the zero locus Z_Q of $\mathrm{Q} = (S, \cdot)$ is the stationary surface of the master action S.

On the other hand, the construction proposed in [6, 5],

$$\{f, g\} = (f, (S, g))\,, \qquad (S, S) = 0\,, \tag{1.1}$$

expresses the Grassmann-even operations through the antibracket and the solution S to the master equation. It was shown there that (1.1) is a Poisson bracket provided one defines it on the subalgebra of functions that are commutative w.r.t. the antibracket. Moreover, the binary operation (1.1)

induces a Lie algebra structure on the space of gauge symmetries in the BV formalism [6] (see also [4]). It was also observed in [4] that (1.1) is a well defined operation in the algebra of smooth functions on the zero locus $\mathrm{Z_Q}$ considered as the quotient of all functions modulo those vanishing on $\mathrm{Z_Q}$.

An important observation of [8, 9] is that (1.1) has an even counterpart. Indeed, one can consider an even Poisson bracket and an odd nilpotent vector field that preserves the Poisson bracket; then the structure (1.1) with the antibracket replaced by the even Poisson bracket determines an antibracket similarly to how (1.1) determines a Poisson bracket. The zero locus of the odd nilpotent Hamiltonian vector field on an even symplectic supermanifold is therefore endowed with an antibracket [7].[1]

Since the zero locus itself is an (odd) Poisson (super)manifold, one can try to make it into a QP manifold. *However, if we start with a general QP manifold*, there are no additional structures that could induce an odd nilpotent vector field *on the zero locus* (since Q vanishes on $\mathrm{Z_Q}$, it induces only the trivial Q-structure on $\mathrm{Z_Q}$). A possible way to make $\mathrm{Z_Q}$ into a QP manifold is to solve the equation

$$(S,S)=0 \quad \mathsf{p}(S)=0, \quad \text{or} \quad (\ \{\Omega,\Omega\}=0 \quad \mathsf{p}(\Omega)=1\ )$$

in either the odd or the even case; this is the generating equation for the odd nilpotent vector fields on $\mathrm{Z_Q}$. In this way, one arrives at various relations between different QP manifolds. This approach is developed in [7], where it is also interpreted in terms of Lie algebra (co)homology.

My claim is that there exists an alternative way to equip the zero locus of Q with an odd nilpotent vector field. This approach is more restrictive but more relevant as regards applications in the BRST quantization. Namely, it can be possible to reduce to the zero locus not only the bracket structure but also the BRST differential. This reduction can be viewed as a procedure relating the BV configuration space and the BFV phase space of the same pure gauge model; the BV antibracket and the BV-quantization BRST differential $\mathrm{Q}=(S,\cdot\,)$ induce the BFV phase space structures (the Poisson bracket and the corresponding BRST differential $\mathrm{Q}_{BFV}=\{\Omega,\cdot\,\}$) on the zero locus of $\mathrm{Q}=(S,\cdot\,)$. In what follows, I will refer to this relation as the *zero locus reduction of the BRST differential*.

2 QP-manifolds

The notion of the QP manifold was introduced in [13, 3] as the geometrical structure underlying the BV quantization. In fact, a similar structure (with the antibracket replaced by the ordinary Poisson bracket) underlies the BFV quantization. I now give a unified definition.

[1] Another interesting feature of the even analogue of (1.1) is that it expresses the odd bracket through the even one and therefore, quantizing the even bracket one can arrive at the quantum counterpart of the antibracket [8]. The quantization amounts to replacing the Poisson brackets in (1.1) with the commutator. We do not discuss this very interesting subject here, however.

Definition 2.1 *A QP manifold is a supermanifold $\mathcal{M}$ equipped with a nondegenerate antibracket (Poisson bracket) $(\,\cdot\,,\,\cdot\,)$ and an odd nilpotent vector field* Q *satisfying*

$$\mathrm{Q}(F,\,G)-(\mathrm{Q}F,\,G)-(-1)^{\mathsf{p}(F)+\mathsf{p}((\cdot,\cdot))}(F,\,\mathrm{Q}G)=0\,,\quad F,G\in\mathcal{F}_{\mathcal{M}}\,,\tag{2.1}$$

with $\mathcal{F}_{\mathcal{M}}$ being the algebra of smooth functions on $\mathcal{M}$ and $\mathsf{p}(\cdot)$ being the Grassmann parity. QP manifolds with a Poisson bracket are called even, and those with an antibracket, odd.

Let $\mathcal{M}$ be a QP manifold. Let also $\mathrm{Z_Q}$ be the zero locus of Q (the submanifold where Q vanishes); this is assumed to be a smooth manifold and Q is required to be regular. This means that the components of Q generates an ideal of functions vanishing on $\mathrm{Z_Q}$. One then has

Proposition 2.1 *[4] The binary operation $\{\,,\,\}:\mathcal{F}_{\mathrm{Z_Q}}\times\mathcal{F}_{\mathrm{Z_Q}}\to\mathcal{F}_{\mathrm{Z_Q}}$*

$$\{f,g\}=(F,QG)|_{\mathrm{Z_Q}}\,,\qquad f,g\in\mathcal{F}_{\mathrm{Z_Q}}\,,\quad F,G\in\mathcal{F}_{\mathcal{M}}\,,\quad F|_{\mathrm{Z_Q}}=f\,,\quad G|_{\mathrm{Z_Q}}=g\,.\tag{2.2}$$

is well defined (i.e., is independent of the choice of the representatives F,G of functions f,g on $\mathrm{Z_Q}$). Moreover, $\{\,,\,\}$ is a Poisson bracket (antibracket) on $\mathrm{Z_Q}$.

Among the QP manifolds, I single out those on which the Q vector field is maximally nondegenerate in the following sense: the matrix $(\frac{\partial}{\partial\Gamma^A}Q^B)|_{p\in Z_Q}$ of the linear operators $\mathrm{Q}_p:T_p\mathcal{M}\to T_p\mathcal{M}$ labelled by $p\in\mathrm{Z_Q}$ should be of the maximal rank at each point $p\in\mathrm{Z_Q}$. Since Q_p is nilpotent, this condition is formalized as follows

Definition 2.2 *A QP manifold $\mathcal{M}$ is called* proper *if the homology of the linear operator $\mathrm{Q}_p:T_p\mathcal{M}\to T_p\mathcal{M}$ is trivial at each point $p\in\mathrm{Z_Q}$.*

In coordinate-free terms, one can define this linear operator Q_p as follows. For $p\in\mathrm{Z_Q}$, one considers the tangent space $T_p\mathcal{M}$ as the quotient of $\mathrm{Vect}_{\mathcal{M}}$ modulo vector fields from $\mathrm{Vect}_{\mathcal{M}}$ that vanish at p. It is easy to see that the operation

$$\mathrm{Q}_p(X|_p)=([\mathrm{Q},\,X])|_p\,,\tag{2.3}$$

is well-defined provided Q vanishes at p. This gives an important property of the Q field on a proper QP manifold:

Proposition 2.2 *Let $\mathcal{M}$ be a proper QP manifold. Then each function f satisfying $\mathrm{Q}f=0$ can be represented as $f=\mathrm{Q}h+\mathrm{const}$ in an appropriately small neighborhood of any point $p\in\mathrm{Z_Q}$.*

In different words, the cohomology of Q is trivial when evaluated in an appropriately small neighborhood of any point $p\in\mathrm{Z_Q}$. If in particular $\mathrm{Q}=(S,\,\cdot\,)$, it follows that

$$S=(\overline{\Omega},S)\tag{2.4}$$

for some (locally defined) function $\overline{\Omega}$ (since S is defined up to a constant, the constant term is omitted here).

Another important property of proper QP manifolds is the following

Proposition 2.3 *[3, 4] Let $\mathcal{M}$ be a proper QP manifold. Then Z_Q is (anti)symplectic. The corresponding Poisson bracket (antibracket) is given by* (2.2).

Remark. All the properties of QP manifolds stated above do hold if the manifold is finite-dimensional. In the infinite dimensional case, however, one should apply these statements with some care. In particular, the standard master action S of e.g., the Yang-Mils theory is a proper solution of the master equation (which is precisely the condition for $\mathrm{Q} = (S, \cdot\,)$ to be proper) but the homology of Q is not trivial when evaluated on the local[2] functionals.

3 Reduction of the BRST differential

Let us be given a topological theory described by the master action S satisfying the classical master equation:

$$(S, S) = 0\,. \tag{3.1}$$

Gauge fixing in the BV framework is described by the anticanonical transformation ψ:

$$S_\Psi = \psi S = e^{(\Psi, \cdot\,)} S, \tag{3.2}$$

where Ψ is a gauge fermion. The BRST transformation acting on the field-antifield space is

$$\mathrm{Q} = (S, \cdot\,). \tag{3.3}$$

I also need the BRST transformation Q_{gf} acting on the gauge-fixing surface (which is identified with $\mathcal{L}_0$ via $\mathcal{L}_\psi = \psi^{-1}\mathcal{L}_0$). In the covariant BV approach, however, there is no natural way to determine the gauge-fixed BRST transformation. Indeed, being defined on the whole field-antifield space $\mathcal{M}$, the BRST transformation does not restrict naturally to the Lagrangian submanifold $\mathcal{L}_\Psi \subset \mathcal{M}$. Thus, in order to determine the gauge-fixed BRST transformation, one needs an additional structure.

Let the field-antifield space $\mathcal{M}$ be the odd cotangent bundle over some "field space" $\mathcal{L}_0$ (this is the case in various applications of the BV quantization). Let also ϕ^A and ϕ^*_A be the coordinates on $\mathcal{L}_0$ and on the fibers (fields and antifields). Then it is possible to determine the gauge-fixed BRST transformation via

$$\mathrm{Q}_{gf} f = (S_\Psi, \pi^* f))|_{\mathcal{L}_0}\,, \qquad h \in \mathcal{F}_{\mathcal{L}_0}\,, \tag{3.4}$$

where π^* is the pullback associated with the canonical projection $\pi\colon \mathcal{M} \to \mathcal{L}_0$ (again, gauge fixing submanifold $\mathcal{L}_\Psi$ is identified with $\mathcal{L}_0$ via $\mathcal{L}_\psi = \psi^{-1}\mathcal{L}_0$).

Unfortunately, this prescription violates the nilpotency of the BRST transformation in general. An important case where Q_{gf} is still nilpotent is where S is linear in the antifields. I will thus assume that S has the form

$$S = S_0(\phi) + \phi^*_A S^A\,. \tag{3.5}$$

[2]here, *local* refers to space-time locality.

Now the gauge-fixed BRST transformation can be represented as

$$\mathrm{Q}_{gf}h = (\mathrm{Q}(\pi^*h))|_{\mathcal{L}_0} = (-1)^{\mathsf{p}(\phi^A)}S^A\frac{\partial}{\partial\phi^A}\,, \tag{3.6}$$

and does not, therefore, depend on the particular choice of the gauge fermion.

Let Z_Q be the zero locus of the odd vector filed $\mathrm{Q} = (S, \cdot\,)$ (the stationary surface of the master action). Assume that there exists an anticanonical transformation φ such that

- $\varphi(\mathcal{L}_0) = \mathrm{Z}_\mathrm{Q}$, i.e., φ transforms the initial Lagrangian submanifold $\mathcal{L}_0$ into Z_Q;
- φ is such that

$$S = (\varphi^*S, \Omega) \tag{3.7}$$

  for some new (in general, locally defined) function Ω; here φ^* is the pullback associated to φ. This can be expressed by saying that transformation φ preserves the property of S to be Q-closed.

The symplectic structure on Z_S is carried over to $\mathcal{L}_0$ by the anticanonical transformation φ. The corresponding Poisson bracket on $\mathcal{L}_0$ reads

$$\{f, g\} = (\pi^*f, (\varphi^*(S), \pi^*g))|_{\mathcal{L}_0}\,, \qquad f, g \in \mathcal{F}_{\mathcal{L}_0}\,. \tag{3.8}$$

Let me now return to master equation (3.1) and rewrite it in terms of the transformed action φ^*S as

$$((\varphi^*S, \Omega), (\varphi^*S, \Omega)) = 0 \quad \Longrightarrow \quad (\varphi^*S, (\Omega, (\varphi^*S, \Omega))) = 0\,.$$

The expression $(\Omega, (\varphi^*S, \Omega))$ formally coincides with the Poisson bracket (3.8) of Ω with itself. To make this correspondence manifest, observe that the equality $(\varphi^*S, X) = 0$ implies that (locally) $X = (\varphi^*S, Y) + \mathrm{const}$ and, thus, $X|_{\mathcal{L}_0} = \mathrm{const}$ provided $\mathcal{L}_0$ is the zero locus of $(\varphi^*S, \cdot\,)$. Now $(\Omega, (\varphi^*S, \Omega))|_{\mathcal{L}_0} = \mathrm{const}$ implies that $(\Omega, (\varphi^*S, \Omega))|_{\mathcal{L}_0} = 0$, since the function Ω is Grassmann-odd. Thus,

$$(\varphi^*S, (\Omega, (\varphi^*S, \Omega))) = 0\,, \quad \Longrightarrow \quad \{\Omega|_{\mathcal{L}_0}, \Omega|_{\mathcal{L}_0}\} = 0\,, \tag{3.9}$$

where $\{\cdot\,,\cdot\}$ is the Poisson bracket (3.8) on $\mathcal{L}_0$.

The crucial observation is that Q_{gf} considered as the vector field on $\mathcal{L}_0$ is generated by the bracket (3.8). Indeed, Eq. (3.7) implies that

$$\mathrm{Q}_{gf}h = (S, \pi^*h)|_{\mathcal{L}_0} = ((\Omega, \varphi^*S), \pi^*h) = \{\Omega|_{\mathcal{L}_0}, h\}\,, \qquad h \in \mathcal{F}_{\mathcal{L}_0}\,,$$

where $\{\cdot\,,\cdot\}$ is the even Poisson bracket (3.8).

Thus, the BV formulation of the topological theory gives rise to all the data of the corresponding BFV scheme, including the Poison bracket and the generating function Ω satisfying $\{\Omega, \Omega\} = 0$. The corresponding BRST transformation $\{\Omega, \cdot\,\}$ coincides with Q_{gf} in the BV framework.

I have to note, however, that this coincidence is rather formal: the main difference between the BV and BFV pictures is that in the BV framework, one works with the *configuration space* (the space of field histories), while in the BFV approach one considers the *phase space*. Further, the BV antibracket is defined on the functionals on *space-time*, and the BFV Poisson bracket is the equal-time bracket defined on the functionals on the space. In Sec. 6, I will give some arguments that make this coincidence exact in several special cases.

4 Existence of φ

The issue of the existence of a globally defined transformation φ is very complicated in the general setting. I will limit myself to a local analysis. The argument is based on Abelianization. It is well known that given a proper solution S to the master equation, one can find new canonical coordinates in some (appropriately chosen) neighborhood, where S becomes quadratic. Specializing this to the case where S has the form $S = \phi^*_A S^A(\phi)$ (since the theory is topological one can assume that the initial action S_0 from (3.5) vanishes) in the initial coordinates, one arrives at the new coordinates $z^\alpha, y^\alpha, z^*_\alpha, y^*_\alpha$ such that

$$S = z^*_\alpha y^\alpha\,, \qquad (z^\alpha, z^*_\beta) = \delta^\alpha_\beta\,, \quad (y^\alpha, y^*_\beta) = \delta^\alpha_\beta\,, \tag{4.1}$$

and the submanifold $\mathcal{L}_0$ is determined by the equations $z^*_\alpha = y^*_\beta = 0$. It is easy to see that in the new coordinates, the stationary surface $\mathrm{Z_Q}$ is determined by the equations $y^\alpha = z^*_\beta = 0$.

One can see that the transformation φ relating the submanifolds $\mathcal{L}_0$ and $\mathrm{Z_Q}$ does not exist in general; the existence of φ implies that the superdimensions of $\mathcal{L}_0$ and $\mathrm{Z_Q}$ are equal. This in turn implies that among the variables x^α and y_α, there are equally many even and odd ones. This is also a sufficient condition for φ to exist, at least locally.

Indeed, let me assume that the condition is fulfilled. Then the variables z, z^*, y, y^* split into the pairs $[x^i, \overline{c}_i]\,, [x^*_i, \overline{c}^i_*]\,, [c^i, b_i]\,, [c^*_i, b^i_*]$ of different parities. Now the master action from (4.1) takes the form

$$S = x^*_i c^i + \overline{c}^i_* b_i\,. \tag{4.2}$$

And the transformation, φ is obviously given by

$$\begin{array}{cccc} \varphi^*(x^i) = x^i & \varphi^*(x^*_i) = x^*_i & \varphi^*(\overline{c}^i_*) = \overline{c}^i_* & \varphi^*(\overline{c}_i) = \overline{c}_i \\ \varphi^*(y^i) = b^i_*, & \varphi^*(b^i_*) = -y^i, & \varphi^*(b_i) = y^*_i, & \varphi^*(y^*_i) = -b_i\,. \end{array} \tag{4.3}$$

This transformation evidently satisfies all the conditions of Sec. 3.

An important case occurs if the master action is constructed via the standard BV prescription for a pure gauge model. Then the required condition always holds provided the configuration space is enlarged by the nonminimal sector (i.e., by the standard auxiliary variables needed for gauge fixing). As we will see in the example in the next section, this is indeed the case with an irreducible pure gauge theory. Thus the nonminimal variables are precisely those which allow one to construct the φ transformation.

5 A simple example

Here, I illustrate the constructions of Sec. 3 in a very simple example of a topological model which is constructed in the BV framework as follows. Let me start with a smooth manifold $\mathcal{X}$ considered as the configuration space of a model with the vanishing action. That the initial action vanishes implies that all the degrees of freedom are pure gauge ones. Let R_α be the corresponding gauge generators, which I assume for simplicity to be linearly independent and form a closed algebra

$$[R_\alpha, R_\beta] = C^\gamma_{\alpha\beta} R_\gamma \tag{5.1}$$

In local coordinates x^i on $\mathcal{X}$, one has $R_\alpha = R^i_\alpha \frac{\partial}{\partial x^i}$. That all degrees of freedom are gauge ones implies that R_α considered as vector fields on $\mathcal{X}$ constitute a basis of the tangent space to $\mathcal{X}$ at each point; equivalently, R^i_α is a nondegenerate matrix.

According to the standard BV quantization prescription, I now introduce the ghost variables c^α for each gauge generator R_α and the antifields variables x^*_i, c^*_α. These variables constitute the so-called *minimal sector* of the gauge theory. The corresponding solution to the classical master equation is

$$S_{min} = x^*_i R^i_\alpha c^\alpha + \frac{1}{2} c^*_\gamma C^\gamma_{\alpha\beta} c^\alpha c^\beta \,. \tag{5.2}$$

To fix the gauge, one should enlarge the minimal set of variables to the *nonminimal one*. The nonminimal variables are the antighosts $\overline{c}_\alpha$, the auxiliary filed b_α, and their conjugate antifields. One should add also the nonminimal terms to the master action. The master action becomes

$$S = S_{min} + \overline{c}^\alpha_* b_\alpha \,. \tag{5.3}$$

which again is a proper solution to the master equation. Now the gauge fixing can be performed by choosing a gauge fermion Ψ.

Let me now concentrate on the zero locus $\mathrm{Z_Q}$ of the master action (5.3). The submanifold $\mathrm{Z_Q}$ is determined by the equations $x^*_i = c^\alpha = \overline{c}^*_\beta = b_\gamma = 0$ and is a Lagrangian submanifold of the configuration space. The Poisson bracket (2.2) on $\mathrm{Z_Q}$ reads

$$\{x^i, c^*_\alpha\} = R^i_\alpha \,, \qquad \{c^*_\alpha, c^*_\beta\} = -C^\gamma_{\alpha\beta} c^*_\gamma \,, \tag{5.4}$$

with all the other brackets vanishing. Thus $\mathrm{Z_Q}$ is equipped with a nondegenerate Poisson bracket. At the same time, the submanifold $\mathcal{L}_0 \in \mathcal{M}$ is equipped with the odd nilpotent vector field Q_{gf} (a gauge fixed BRST transformation (3.6)); in the local coordinates $x, \pi, c, \overline{c}$, it reads

$$\mathrm{Q}_{gf} = c^\alpha R^i_\alpha \frac{\partial}{\partial x^i} - \frac{1}{2} C^\gamma_{\alpha\beta} c^\alpha c^\beta \frac{\partial}{\partial c^\gamma} - \pi_\alpha \frac{\partial}{\partial \overline{c}_\alpha} \,. \tag{5.5}$$

Let me now turn to the construction of the anticanonical transformation φ relating the Lagrangian submanifolds $\mathrm{Z_Q}$ and $\mathcal{L}_0$. For the Abelian theory (i.e., with the structure constants C^k_{ij} vanishing), the required transformation is simply an obvious modification of transformation (4.3).

In the nonabelian case, however, one should add additional terms. Consider the following generalization of (4.3)

$$\begin{gathered}\varphi^*(x^i) = x^i \quad \varphi^*(x_i^*) = x_i^*\,, \quad \varphi^*(c^\alpha) = b_*^\alpha\,, \quad \varphi^*(b_*^\alpha) = -c^\alpha\,,\\ \varphi^*(b_\alpha) = c_\alpha^* - C_{\alpha\beta}^\gamma \overline{c}_\gamma b_*^\beta\,, \quad \varphi^*(c_\alpha^*) = -b_\alpha - C_{\alpha\beta}^\gamma \overline{c}_\gamma c^\beta\,,\\ \varphi^*(\overline{c}_\alpha) = \overline{c}_\alpha\,, \quad \varphi^*(\overline{c}_*^\alpha) = \overline{c}_*^\alpha + C_{\gamma\beta}^\alpha b_*^\gamma c^\beta\,.\end{gathered} \tag{5.6}$$

We have

Proposition 5.1 *Transformation* (5.6) *is anticanonical (preserves the antibracket), is globally defined, and satisfies the condition*

$$S = (\Omega, \varphi^* S)\,. \tag{5.7}$$

Indeed, transformation (5.6) is globally defined by construction (since it preserves the base $\mathcal{X}$). The transformed master action is

$$\varphi^* S = x_i^* R_\alpha^i b_*^\alpha - \frac{1}{2} b_\gamma C_{\alpha\beta}^\gamma b_*^\alpha b_*^\beta + \overline{c}_*^\alpha c_\alpha^* - \overline{c}_*^\alpha C_{\alpha\beta}^\gamma \overline{c}_\gamma b_*^\beta - c_\gamma^* C_{\alpha\beta}^\gamma c^\beta b_*^\alpha\,. \tag{5.8}$$

It is easy to see by direct computations that

$$\Omega = c^\alpha b_\alpha - \frac{1}{2} \overline{c}_\gamma C_{\alpha\beta}^\gamma c^\alpha c^\beta\,. \tag{5.9}$$

solves the condition (5.7).

It now follows from the general treatment of section 3 that $\mathcal{L}_0$ is equipped with Poisson bracket (3.8). In the local coordinates on $\mathcal{L}_0$, we have

$$\begin{gathered}\{x^i, b_\alpha\} = -R_\alpha^i\,, \quad \{b_\alpha, b_\beta\} = C_{\alpha\beta}^\gamma b_\gamma\,,\\ \{\overline{c}_\alpha, c^\beta\} = -\delta_\alpha^\beta\,, \quad \{\overline{c}_\alpha, b_\beta\} = C_{\alpha\beta}^\gamma \overline{c}_\gamma\,, \quad \{c^\alpha, b_\beta\} = C_{\beta\gamma}^\alpha c^\gamma\,.\end{gathered} \tag{5.10}$$

The Jacobi identity for bracket (5.10) holds provided $\varphi^* S$ satisfies the classical master equation.

It also follows from the results of Sec. 3 that Q_{gf} given by (5.5) is compatible with Poisson bracket (5.10). Moreover,

$$Q_{gf} = \{\Omega, \cdot\} \tag{5.11}$$

Thus, at least at the formal level, we obtain all the objects of the BFV quantization: the Poisson bracket and the generating function (the BRST charge) Ω. To make the coincidence more clear let me choose new coordinates

$$q^i = x^i\,, \quad p_i = -R_i^\alpha (b_\alpha + C_{\alpha\beta}^\gamma \overline{c}_\gamma c^\beta)\,, \quad \overline{\mathcal{P}}_\alpha = -\overline{c}_\alpha\,, \tag{5.12}$$

with all the others being unchanged. These are the canonical (Darboux) coordinates for the Poisson bracket (5.10)

$$\{x^i, p_j\} = \delta_j^i\,, \quad \{c^\alpha, \overline{\mathcal{P}}_\beta\} = \delta_\beta^\alpha\,. \tag{5.13}$$

In the new coordinate system, one gets

$$\Omega = c^\alpha T_\alpha - \frac{1}{2}\overline{\mathcal{P}}_\gamma C^\gamma_{\alpha\beta} c^\alpha c^\beta \,, \qquad T_\alpha = -R^i_\alpha p_i \,. \tag{5.14}$$

which is the standard form of the BRST charge for the pure gauge theory with the first-class constraints given by T_α. Moreover, at the formal level,[3] this is the BRST charge of the same theory. Indeed, the first class constraints T_α satisfy the same Lie algebra as the gauge transformation R_α in the BV formulation; the phase space of this system is the cotangent bundle $T^*\mathcal{X}$ enlarged by the ghosts c^α and the corresponding momenta $\overline{\mathcal{P}}_\alpha$.

A remarkable feature of the proposed reduction procedure is that it reduces the BV scheme of the topological model in the *nonminimal* sector to the corresponding BFV scheme in the *minimal* sector. Moreover, the zero locus reduction of the BRST differential identifies the nonminimal variables in the BV quantization with the momenta associated to the minimal variables in the BFV picture of the same theory.

6 A Generalization

Until this point, I discussed a finite dimensional analogue of the topological theory in the BV and BFV framework. I would now like to give some arguments showing that the zero locus reduction works well not only at the formal level. For simplicity, I consider only the 1-dimensional field theory, or quantum mechanics. To avoid considerable technical complications, let me also assume that the topological theory at hand can be represented in the form where the master action and the gauge generators come from the corresponding objects in the target space. This means that

$$S = \int dt \; s(\phi(t), \phi^*(t)) \,, \qquad R^i_\alpha(t,t') = \delta(t-t') r^i_\alpha(\phi(t)) \,, \tag{6.1}$$

where s and r_α are functions and vector fields on the target space (which is also antisymplectic). This is the case for a wide class of pure gauge theories including e.g., SQM, Chern–Simons model, and topological sigma models (see [3] for details).

Then the BV configurations space is the space of smooth maps from the "time line" into the target space. The configuration space is equipped with the antisymplectic structure given by the lift of the antisymplectic structure in the target space to the space of maps into the target space (see [3]).

Since the zero locus reduction can be performed in the target space, one arrives at the symplectic submanifold $\mathcal{L}_0$ equipped with the odd vector field $Q_{gf} = \{\Omega, \cdot\}$. The space of maps from the time line into the symplectic manifold can be identified with the space of Hamiltonian trajectories. Under this identification, the Poisson bracket in the target space becomes the equal-time Poisson bracket. One thus arrives at the BFV formulation of the same topological model.

[3] An account is not yet taken of an explicit time dependence of the Hamiltonian scheme. This issue will be explained in Sec. 6

7 The BV quantization from the BFV approach.

Similar considerations can be applied starting with the BFV formulation of a topological model. In that case, the zero locus of the BFV BRST differential $\mathrm{Q} = \{\Omega, \cdot\}$ is equipped with a nondegenerate antibracket. A similar reduction procedure (where one should find the transformation relating the zero locus of Q and the submanifold of coordinates) provides this submanifold with an odd Hamiltonian vector field Q, which can be interpreted as the BRST transformation generated by some (Grassmann-even) function S viewed as the master action in the corresponding BV formulation. Again, the BFV formulation in the nonminimal sector reduces to the BV formulation in the minimal sector.

8 Conclusions

We have explicitly constructed the BFV phase space of a pure gauge (topological) model starting with the BV configuration space of the BV formulation of the same model. The construction is based on two important ingredients.

The first one is the even Poisson structure on the zero locus of the BRST differential (stationary surface of the master action) in the BV formalism [3, 4].

The second is the construction of the appropriate anticanonical transformation φ relating the gauge fixing surface (which in our simplified approach is identified with the initial submanifold $\mathcal{L}_0$ via the gauge fixing transformation) and the stationary surface $\mathrm{Z_Q}$ of the master action. The transformation φ allows us to identify $\mathrm{Z_Q}$ with $\mathcal{L}_0$ and, thus, to carry over the gauge fixed BRST transformation Q_{gf} defined on $\mathcal{L}_0$ to a Hamiltonian vector field on $\mathrm{Z_Q}$ (equivalently, to carry over the Poisson bracket from $\mathrm{Z_Q}$ to $\mathcal{L}_0$; this is the viewpoint adopted in the paper). In this way, one arrives at the Lagrangian submanifold equipped with two compatible structures, the even Poisson bracket and the odd nilpotent vector filed. A further analysis shows that one can formally identify this Lagrangian submanifold with the phase space of the corresponding BFV phase space.

An important feature of the proposed construction is that the BV variables of the nonminimal sector play the role of momenta that are conjugate to the variables of the BFV minimal sector. This gives an interesting geometrical interpretation of the nonminimal variables.

It should be noted that the relation discussed here is essentially based on the fact that the initial BV structures correspond to the topological theory. However, in the general gauge theory setting, one can formally separate the physical sector and the pure gauge sector and then perform the reduction in the pure gauge sector. This suggests an interpretation of the results presented here as those describing the structure of the pure gauge sector of the general gauge theory.

This also suggests that the Poisson bracket in the BFV formulation of the general gauge theory splits into two parts: the standard Poisson bracket in the physical sector and the Poisson bracket in the pure gauge sector. The latter is related to the BV antibracket from the corresponding BV formulation via the present construction. At least for an irreducible theory with a closed gauge algebra, this relation originates in the fact that both the BV antibracket in the pure gauge sector

and the BFV Poisson bracket in the pure gauge sector come from a single Gerstenhaber-like bracket structure in the gauge algebra complex (see [7] for the details). This is somewhat reminiscent of the results of [10], where the general relationship was established between certain Lie algebras w.r.t. the BV antibracket and their Hamiltonian counterparts w.r.t. the BFV Poisson bracket.

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