Surface observables and the Weyl anomaly

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Abstract

I review the computation of the conformal anomaly of a Wilson surface observable in free two-form gauge theory in six dimensions.

M-theory admits configurations with some number N of coincident five-branes embedded in a curved eleven-dimensional space. Small fluctuations around this configuration are described by a $(0,2)$ superconformal field theory defined on the six-dimensional world-volume M of the branes, and with a background metric $G_{\mu\nu}$ given by the embedding. There is a conformally invariant observable $W(\Sigma)$ associated with every two-cycle Σ in M. When $N=1$, this theory is simply given by a free $(0,2)$ tensor multiplet, i.e. a two-form $B_{\mu\nu}$ with self-dual field strength, eight spinors ψ^i and five scalars ϕ^a, and the observable associated with a two-cycle Σ is given by the exponential of the integral of $B_{\mu\nu}$ over Σ:

$$\langle W(\Sigma)\rangle = \left\langle \exp 2\pi i \int_\Sigma B \right\rangle . \tag{1}$$

For finite N, the theory is a difficult to study interacting superconformal field theory, about which rather little is known. For large N, however, there is a 'holographic' description by a weakly coupled supergravity theory on an open seven-manifold X such that $\partial X = M$. The expectation value of the observable associated with a two-cycle Σ is then given by the exponential of the volume of an embedded minimal open three-manifold D such that $\partial D = \Sigma$:

$$\langle W(\Sigma)\rangle = \exp 2\pi i \int_D V_D, \tag{2}$$

where V_D is the volume form on D.

The theory itself, and also the observable $W(\Sigma)$ for an arbitrary two-cycle Σ, are invariant under conformal rescalings of the background metric $G_{\mu\nu}$. Formally, the quantity

$$I = -\frac{1}{4\pi^2} \log \langle W(\Sigma)\rangle \tag{3}$$

is therefore conformally invariant. However, I diverges and needs to be regularized. By adding local counterterms, we can cancel the divergences and obtain a finite I_{fin}. This procedure can be carried out in a manifestly covariant way, but conformal invariance is in general lost. Under a conformal transformation

$$\delta G_{\mu\nu} = 2\phi G_{\mu\nu} \tag{4}$$

with infinitesimal parameter function ϕ, I_{fin} transforms as

$$\delta I_{\text{fin}} = -4\pi^2 \int_\Sigma d^2\sigma \sqrt{g} \mathcal{A}, \tag{5}$$

where g is the determinant of the induced metric $g_{\alpha\beta}$ on Σ, and the conformal or Weyl anomaly $\mathcal{A}$ is given by some local expression linear in ϕ.

By the Wess-Zumino consistency conditions, the conformal variation of the anomaly $\mathcal{A}$ is a total derivative. It follows that $\mathcal{A}$ must be a linear combination of the terms

$$\begin{aligned} & R_{(2)}\phi \\ & \left(G_{\mu\nu}\nabla^2 X^\mu \nabla^2 X^\nu - 4 g^{\alpha\beta} P_{\alpha\beta}\right)\phi \\ & g^{\alpha\gamma} g^{\beta\delta} W_{\alpha\beta\gamma\delta}\phi \\ & \nabla^2 X^\mu D_\mu \phi, \end{aligned} \tag{6}$$

where $R_{(2)}$ is the curvature scalar of the induced metric $g_{\alpha\beta}$ on Σ, $\nabla^2 X^\mu$ is the mean curvature vector of the embedding of Σ in M, and $P_{\alpha\beta}$ and $W_{\alpha\beta\gamma\delta}$ are the pullbacks of the tensor

$$P_{\mu\nu} = \frac{1}{4}\left(R_{\mu\nu} - \frac{1}{10} R G_{\mu\nu}\right) \tag{7}$$

and the Weyl tensor $W_{\mu\nu\rho\sigma}$ respectively to Σ.

For large N, the anomaly $\mathcal{A}$ was calculated in [1] by generalizing the computation in [2] with the result

$$\mathcal{A} \sim \left(G_{\mu\nu}\nabla^2 X^\mu \nabla^2 X^\nu - 4g^{\alpha\beta}P_{\alpha\beta}\right)\phi - 2\nabla^2 X^\mu D_\mu \phi. \tag{8}$$

In this talk, I will review the calculation of $\mathcal{A}$ for $N = 1$ but without any self-duality condition imposed on the field strength of $B_{\mu\nu}$. For more details, I refer to our paper [3].

The gauge fixed action for $B_{\mu\nu}$ is

$$S = \int_M d^6x \sqrt{G} G^{\lambda\rho} G^{\mu\sigma} G^{\nu\tau} \left(-\frac{3}{4} D_{[\lambda} B_{\mu\nu]} D_\rho B_{\sigma\tau} - \frac{\alpha}{2} D_\lambda B_{\rho\mu} D_\nu B_{\tau\sigma}\right) \tag{9}$$

with an arbitrary non-zero constant α multiplying the gauge fixing term. Apart from this term, the action is conformally invariant, which means that the conformal variation of the propagator $\Delta_{\mu\nu;\rho\sigma}(X, X')$ is a total derivative. The same is true for its pullback $\Delta_{\alpha\beta;\gamma\delta}(\sigma, \hat{\sigma})$ to an embedded two-cycle Σ, i.e.

$$\delta\Delta_{\alpha\beta;\gamma\delta}(\sigma, \hat{\sigma}) = \partial_{[\alpha}\Lambda_{\beta];\gamma\delta}(\sigma, \hat{\sigma}) + \hat{\partial}_{[\gamma}\Lambda'_{|\alpha\beta|;\delta]}(\sigma, \hat{\sigma}) \tag{10}$$

for some $\Lambda_{\beta;\gamma\delta}(\sigma, \hat{\sigma})$ and $\Lambda'_{\alpha\beta;\delta}(\sigma, \hat{\sigma})$ that are linear in the parameter ϕ of a conformal variation of the metric. It follows that the quantity

$$I = \int_\Sigma d\sigma^\alpha \wedge d\sigma^\beta \int_\Sigma d\hat{\sigma}^\gamma \wedge d\hat{\sigma}^\delta \Delta_{\alpha\beta;\gamma\delta}(\sigma, \hat{\sigma}) \tag{11}$$

is formally invariant under a conformal transformation.

However, $\Delta_{\alpha\beta;\gamma\delta}(\sigma, \hat{\sigma})$ diverges along the diagonal of $\Sigma \times \Sigma$, so I is not really well defined. We therefore consider the regulated quantity

$$I_\epsilon = \int_\Sigma d\sigma^\alpha \wedge d\sigma^\beta \int_\Sigma d\hat{\sigma}^\gamma \wedge d\hat{\sigma}^\delta \Theta(s^2(\sigma, \hat{\sigma}) - \epsilon^2)\Delta_{\alpha\beta;\gamma\delta}(\sigma, \hat{\sigma}), \tag{12}$$

where $\Theta(t)$ is the step function, $s^2(\sigma, \hat{\sigma})$ is the square of the geodesic distance and $\epsilon > 0$ is a cutoff. We expect that

$$I_\epsilon = \epsilon^{-2} I_2 + \log\epsilon\, I_0 + I_{\text{fin}} + \mathcal{O}(\epsilon), \tag{13}$$

where the coefficients I_2 and I_0 of the divergent terms are given by some local expressions that can be cancelled by counterterms so that we are left with the finite quantity I_{fin}. This regularization breaks conformal invariance, and under a conformal variation we expect that

$$\delta I_\epsilon = \epsilon^{-2}\mathcal{A}_2 + \mathcal{A} + \mathcal{O}(\epsilon), \tag{14}$$

where $\mathcal{A} = \delta I_{\text{fin}}$ is the conformal anomaly. On the other hand, we have that

$$\begin{aligned}\delta I_\epsilon \;=\; & \int_\Sigma d\sigma^\alpha \wedge d\sigma^\beta \int_\Sigma d\hat{\sigma}^\gamma \wedge d\hat{\sigma}^\delta \delta(s^2(\sigma, \hat{\sigma}) - \epsilon^2) \\ & \left(\delta s^2(\sigma, \hat{\sigma})\Delta_{\alpha\beta;\gamma\delta}(\sigma, \hat{\sigma}) - \partial_\alpha s^2(\sigma, \hat{\sigma})\Lambda_{\beta;\gamma\delta}(\sigma, \hat{\sigma}) - \hat{\partial}_\gamma s^2(\sigma, \hat{\sigma})\Lambda'_{\alpha\beta;\delta}(\sigma, \hat{\sigma})\right),\end{aligned} \tag{15}$$

where $\delta s^2(\sigma, \hat{\sigma})$ denotes the conformal variation of the geodesic distance.

We see from the above that δI_ϵ, and thus the anomaly, can be calculated just from a knowledge of the short distance expansion of the propagator, the geodesic distance and their conformal variations. In Riemann normal coordinates around X' and with $X^\mu = X'^\mu + \Delta X^\mu$, these are given by

$$\Delta_{\rho\sigma;\mu\nu}(X, X') \;=\; -\frac{1}{4\pi^3}|\Delta X|^{-4}\left[\eta_{\rho\mu}\eta_{\sigma\nu} - \frac{1}{3}\left(P_{\mu\kappa}\eta_{\lambda\rho}\eta_{\sigma\nu} + \eta_{\mu\kappa}P_{\lambda\rho}\eta_{\sigma\nu}\right)\Delta X^\kappa \Delta X^\lambda\right.$$

$$
\begin{aligned}
& + \left(\frac{4}{3}P_{\rho\mu}\eta_{\sigma\nu} - \frac{1}{2}W_{\rho\mu\sigma\nu}\right)|\Delta X|^2 + \mathcal{O}(\Delta X^3)\Big] \\
\delta\Delta_{\rho\sigma;\mu\nu}(X,X') \;=\; & -\frac{1}{4\pi^3}\partial_\rho|\Delta X|^{-2}\left(\eta_{\sigma\nu}\partial_\mu\phi + \frac{1}{2}\Delta X^\tau\eta_{\sigma\nu}\partial_\mu\partial_\tau\phi + \mathcal{O}(\Delta X^2)\right) \\
& -\frac{1}{4\pi^3}\partial'_\mu|\Delta X|^{-2}\left(\eta_{\nu\sigma}\partial_\rho\phi + \frac{1}{2}\Delta X^\tau\eta_{\nu\sigma}\partial_\rho\partial_\tau\phi + \mathcal{O}(\Delta X^2)\right) \qquad (16)
\end{aligned}
$$

antisymmetrized in $\rho\sigma$ and $\mu\nu$ with the tensors evaluated at X', and

$$
\begin{aligned}
s^2(X,X') &= |\Delta X|^2 + \mathcal{O}(\Delta X^5) \\
\delta s^2(X,X') &= |\Delta X|^2\left(2\phi + \partial_\mu\phi\Delta X^\mu + \frac{1}{3}\partial_\mu\partial_\nu\phi\Delta X^\mu\Delta X^\nu\right) + \mathcal{O}(\Delta X^5). \qquad (17)
\end{aligned}
$$

respectively.

The computations are most easily performed by introducing Riemann normal coordinates (with respect to the induced metric) on Σ as well. The delta function in (15) reduces the integral over $d^2\hat{\sigma}$ to a one-dimensional integral, which is elementary. We are then left with an expression integrated over $d^2\sigma$ only. The end result is that

$$
\delta I_\epsilon = -\frac{1}{4\pi^2}\int_\Sigma d^2\sigma\sqrt{g}\left(\epsilon^{-2}4\phi + \mathcal{A} + \mathcal{O}(\epsilon)\right), \qquad (18)
$$

where the anomaly $\mathcal{A}$ is given by

$$
\mathcal{A} = \left(-\frac{3}{4}(\nabla^2X^\mu\nabla^2X^\nu G_{\mu\nu} - 4g^{\alpha\beta}P_{\alpha\beta}) - \frac{1}{2}R_{(2)} - \frac{1}{6}g^{\alpha\gamma}g^{\beta\delta}W_{\alpha\beta\gamma\delta}\right)\phi - \frac{5}{6}\nabla^2X^\mu D_\mu\phi. \qquad (19)
$$

We note that the quadratic divergence indeed can be cancelled by a local counterterm, which in fact is just the area of Σ. There is also a logarithmic divergence in I_ϵ (but not in δI_ϵ) which is likewise given by a local expression. Finally, we point out that $\mathcal{A}$ is *not* proportional to the large N anomaly computed in [1].

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References

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