The Model of the Nonlinear Duality Symmetrical Electrodynamics without Sources

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Abstract

New essentially nonlinear model of classical electrodynamics with duality symmetrical Lagrangian are proposed. The theory turned out to possess unexpected extra discrete duality symmetry. The equations of motion are invariant under some transformation of strengths accompanied by transformation of potentials.

1 The Duality Transformations

In the modern form the duality transformations were introduced by Rainich, and Misner and Wheeler [1]

$$F'_{\mu\nu} = F_{\mu\nu}\cos\theta + *F_{\mu\nu}\sin\theta \equiv e^{*\theta}F_{\mu\nu}$$

$$*F'_{\mu\nu} = -F_{\mu\nu}\sin\theta + *F_{\mu\nu}\cos\theta \equiv e^{*\theta} * F_{\mu\nu} \tag{1}$$

where $*F_{\mu\nu} = 1/2\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$.

In view of formal similarity of the duality conjugation operator and imaginary unit and because of the fact that the transition operator from $\mathbf{F}$ to $\mathbf{F}'$ looks like a rotation in a plane, it becomes evident that transformation (1) is referred to as the duality rotation.

The pure Maxwell electrodynamics equations with conventional quadratic Lagrangian

$$\mathcal{L} = gF^{\mu\nu}F_{\mu\nu} \equiv g\mathbf{F}^2$$

prove to be invariant under the transformations (1), while the Lagrangian itself does not possess the invariance! There is no symmetry but it indicated that the symmetry might be realized nonlinearly (cf. spontaneous symmetry breaking). This rather unusual fact, being well known, hardly attracted due attention of the theorists for a long time. And just recently attempts to generalize *electric-magnetic duality* appeared. There was a number of attempts to create a *quadratic* Lagrangian with the duality symmetry (1). Main problem to solve was Lorentz invariance. See, e.g. [2, 3, 4, 5, 6, 7, 8]

It would be quite in spirit of the modern approaches to consider the Lagrangian with the explicit duality symmetry [9]. The only duality invariant is known to be in the pure Maxwell electrodynamics which is also the Lorentz one. It has not been considered (as far as I know) as a Lagrangian what is quite clear – it would lead to the badly nonlinear motion equations. Nevertheless, it is of interest to consider a model of pure electrodynamics with dual-invariant Lagrangian, first, in virtue of the unusual and very nice properties of the model and, second, this essentially nonlinear theory may be related in some way both to Maxwell electrodynamics and to magnetic charge problem.

One can obtain the only duality invariant expression as follows. Let us consider a complex quadratic form

$$\mathbf{F}^2 + i * \mathbf{F}\mathbf{F}$$

This form is usually regarded in complexifing $\mathbf{F}$-space. It is invariant under the complex orthogonal transformations $O(3,C)$. With respect to the duality group (1) the form transforms as follows

$$\mathbf{F}^2 + i * \mathbf{F}\mathbf{F} = (\mathbf{F}'^2 + i * \mathbf{F}'\mathbf{F}')e^{2i\theta}$$

This formula, first, establishes isomorphism between the duality rotations and the ordinary phase transformations. Second, it makes clear that transformations (1) are not a subgroup of $O(3,C)$. The complex orthogonal transformations should not mix up with the duality ones. Now it is quite clear that

$$(\mathbf{F}^2)^2 + (*\mathbf{F}\mathbf{F})^2 = inv$$

It is the only expression which is invariant under both Lorentz and the duality transformations. It should be noted also that for any component $F_{\mu\nu} = f$

$$f + i * f = (f' + i * f')e^{i\theta} \quad \longrightarrow f^2 + *f^2 = inv$$

Let the Lagrangian of the pure duality electrodynamics be chosen as follows

$$\mathcal{L} = g\sqrt{(\mathbf{F}^2)^2 + (*\mathbf{FF})^2}$$

Variational principle yields the equations

$$\left(F^{\mu\nu} \frac{\mathbf{F}^2}{\sqrt{(\mathbf{F}^2)^2 + (*\mathbf{FF})^2}} + *F^{\mu\nu} \frac{*\mathbf{FF}}{\sqrt{(\mathbf{F}^2)^2 + (*\mathbf{FF})^2}} \right),_\nu = 0$$

It is seen that the local angle φ determined by electromagnetic strength at the point of space-time appears.

$$\cos\varphi = \frac{\mathbf{F}^2}{\sqrt{(\mathbf{F}^2)^2 + (*\mathbf{FF})^2}}, \quad \sin\varphi = \frac{*\mathbf{FF}}{\sqrt{(\mathbf{F}^2)^2 + (*\mathbf{FF})^2}}$$

And now the equations of motion can be written in a shorter form

$$(F^{\mu\nu}\cos\varphi + *F^{\mu\nu}\sin\varphi),_\nu = 0$$

Introducing notation

$$F^{\mu\nu}\cos\varphi + *F^{\mu\nu}\sin\varphi = e^{*\varphi(x)}F^{\mu\nu}$$

one can rewrite these equations in a more elegant way

$$(e^{*\varphi(x)}F^{\mu\nu}),_\nu = 0, \tag{2}$$

Adding the identity

$$*F^{\mu\nu}{}_{,\nu} \equiv 0 \tag{3}$$

we obtaine the complete duality invariant system of equations (2),(3) which is essentially non-linear.

2 Transformation Properties of the Model under the Duality Group

Let us write transformation of some objects.

$$F_{\mu\nu} = e^{-*\theta}F'_{\mu\nu}, \quad *F_{\mu\nu} = e^{-*\theta} * F'_{\mu\nu} \tag{4}$$

$$\cos\varphi = \cos(\varphi' + 2\theta), \quad \sin\varphi = \sin(\varphi' + 2\theta)$$

$$e^{*\varphi}F_{\mu\nu} = e^{*\theta}(e^{*\varphi'}F'\mu\nu) \tag{5}$$

All these formulas can be obtained directly. We can see that the left-hand sides of the system (2),(3) transform in a different way. The left-hand side of (3) transforms like (4), i.e., *co-gradiently* with respect to the field $F_{\mu\nu}$. The left-hand side of (2), as is seen from (5), transforms by means of invert transformation, i. e. *contra-gradiently* with respect to $F_{\mu\nu}$

3 Duality Electrodynamics as a Maxwell System with Source

One can write the set (2),(3) in a different form resolving the first of the equations with respect to $F^{\mu\nu}{}_{,\nu}$

$$F^{\mu\nu}{}_{,\nu} = \varphi_{,\nu}\,(F^{\mu\nu}\tan\varphi - *F^{\mu\nu}) \equiv j^\mu \tag{6}$$

$$*F^{\mu\nu}{}_{,\nu} \equiv 0$$

One can consider such representation of (2),(3) as a way to break, the symmetry since both sides of (6) transform in the different way. The equation (6) takes the form of the Maxwell equation with a source. Let us suppose that the nonlinear set (6) has a solution of the form

$$F_{0\mu} = \mathbf{E} \sim \frac{1}{r}, \qquad j^0 = o(\frac{1}{r}), \quad r \to \infty$$

Then, perhaps, the notation of the right-hand side of (6) as a current could be justified. It shoud be noted that genuine static central-symmetrical field, e.g. Coulomb's one, makes the right-hand side of (6) equal zero. But it would be better to find an *asymptotically* central symmetrical solution.

It is interesting to notice that the pure Coulomb field as well as any superposition of such fields satisfy the equations (2),(3) . This means that equations (2),(3) are the example of a nonlinear equations for which the superposition principle is valid!

It should be noted that quantization of charge which is effective in such a theory would arise as a result of quantising nonlinear field theory.

4 Instantons in Electrodynamics

Instantons in electrodynamics are known to be absent because of topology triviality of the gauge group . The term "instanton" should be made more precise.

Let us note that equations of motion are satisfied if the condition

$$F_{\mu\nu} \cos\varphi + *F_{\mu\nu} \sin\varphi = 0 \tag{7}$$

is fulfilled. Let it be called as the generalized instanton equation. The term instanton, will be made clear later when discussing an analogous equation in the non-abelian theory. It is convenient to denote $a = \mathbf{F}^2$, $b = *\mathbf{F}\mathbf{F}$. Then projecting (7) onto $\mathbf{F}$, one obtains $a^2 + b^2 = 0$, or, because of reality of the fields $*\mathbf{F}, \mathbf{F}$: $a = 0$, $b = 0$. Thus, the electrodynamical instantons are e.g. plane-waves.

5 The Local Duality Transformations

The appearance of an angle $\varphi(x)$ in a natural manner suggests to introduce the space-time point dependent duality transformations. So

$$\tilde{F}^{\mu\nu} = e^{*\varphi} F^{\mu\nu} \tag{8}$$

To go further, it is necessary to know what is $*\tilde{\mathbf{F}}$? The definition is as follows. For any function f given on $\mathbf{F}$-space, the duality conjugation procedure is

$$*f(\mathbf{F}) = f(*\mathbf{F}) \tag{9}$$

Then, for example,

$$*\tilde{F}_{\mu\nu}(\mathbf{F}) = \tilde{F}_{\mu\nu}(*\mathbf{F}) \tag{10}$$

Noticing that

$$\cos\varphi(*\mathbf{F}) = -\cos\varphi, \qquad \sin\varphi(*\mathbf{F}) = -\sin\varphi$$

we find by means of (10)

$$*\tilde{F}_{\mu\nu} = F_{\mu\nu}\sin\varphi - *F_{\mu\nu}\cos\varphi = -e^{*\varphi} * F_{\mu\nu} \tag{11}$$

Now may be easily proved the properties

$$(\tilde{\mathbf{F}}^2)^2 + (*\tilde{\mathbf{F}}\tilde{\mathbf{F}})^2 = (\mathbf{F}^2)^2 + (*\mathbf{F}\mathbf{F})^2$$

$$\cos\varphi = \cos\tilde{\varphi}, \qquad \sin\varphi = \sin\tilde{\varphi} \tag{12}$$

For instance

$$\cos\tilde{\varphi} = \frac{\tilde{\mathbf{F}}^2}{\sqrt{(\tilde{\mathbf{F}}^2)^2 + (*\tilde{\mathbf{F}}\tilde{\mathbf{F}})^2}} =$$

$$= \frac{1}{\sqrt{(\mathbf{F}^2)^2 + (*\mathbf{FF})^2}} \left(\mathbf{F}^2(\cos^2\varphi - \sin^2\varphi) + 2(*\mathbf{FF})\sin\varphi\cos\varphi\right) =$$
$$= \cos\varphi\cos 2\varphi + \sin\varphi\sin 2\varphi = \cos\varphi$$

Taking into account (9),(10),(12) one can invert the formulas (8),(11)

$$F_{\mu\nu} = e^{*\varphi}\tilde{F}_{\mu\nu} = e^{*\tilde{\varphi}}\tilde{F}_{\mu\nu}, \qquad *F_{\mu\nu} = -e^{*\varphi} * \tilde{F}_{\mu\nu} = -e^{*\tilde{\varphi}} * \tilde{F}_{\mu\nu} \tag{13}$$

Now it the system (2),(3) can be written in terms of the tensor $\tilde{F}_{\mu\nu}$

$$\tilde{F}^{\mu\nu}{}_{,\nu} = 0, \qquad \left(e^{*\tilde{\varphi}} * \tilde{F}^{\mu\nu}\right)_{,\nu} \equiv 0$$

and like the set (6) can be written in the form

$$\tilde{F}^{\mu\nu}{}_{,\nu} = 0, \qquad *\tilde{F}^{\mu\nu}{}_{,\nu} \equiv \tilde{\jmath}^{\mu}$$

where $\jmath^{\mu}$ is formally a magnetic charge current.

However, the interpretation of the equations turned out to be not correct. I am going to show that the transformation in question is in effect a symmetry. First, let us change notation:

$$\tilde{F}_{\mu\nu} = *f_{\mu\nu}$$

Then in terms of $f_{\mu\nu}$ the equations of motion are given by

$$*f^{\mu\nu}{}_{,\nu} = 0, \qquad (e^{*\varphi_f} f^{\mu\nu})_{,\nu} \equiv 0$$

Next let make transformation of electromagnetic potentials:

$$A_\mu \longmapsto a_\mu : \qquad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$$

To be more precise, it is a transformation to new potentials, not transformation, for connection a_μ and A_M which is nonlocal and could not be written down manifestly .

Finally, the equations of motion take the form

$$*f^{\mu\nu}{}_{,\nu} \equiv 0, \qquad (e^{*\varphi_f} f^{\mu\nu})_{,\nu} = 0$$

and coincide with the original ones:

$$(e^{*\varphi(x)}F^{\mu\nu})_{,\nu} = 0, \qquad *F^{\mu\nu}{}_{,\nu} \equiv 0 \tag{14}$$

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