Instanton propagator in the scalar model: exact expression and contribution to instanton induced processes

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Abstract

The propagator in the instanton background in the $(-\lambda\phi^4)$ scalar model in four dimensions is studied. Leading and sub-leading terms of its asymptotics for large momenta and its on-shell double residue are calculated. These results are applied to the analysis of the initial state and initial-final state corrections and the calculation of the next-to-leading (propagator) correction to the exponent of the cross section of multiparticle scattering processes.

1 Introduction

In theories with non-trivial structure of vacua a number of interesting physical effects, induced by instanton solutions, appear. In the present article we will study shadow processes [1]. These are non-perturbative processes in which both the initial and the final state are in the false vacuum. Apart from standard perturbative contributions, the processes which start and end in the false vacuum acquire additional contributions due to the underbarrier tunelling of the system to another vacuum and its return to the initial one. This transition is obviously induced by an instanton solution and goes through the intermediate state containing a bubble of the true vacuum. We would like to mention that other examples of instanton induced processes are transitions with baryon number violation between the vacua in the electroweak theory [2] and the decay of a metastable (false) vacuum due to underbarrier tunelling from a false vacuum to the true one [3].

Much work has been done to study the instanton induced transitions, and quite effective techniques for the calculation of the probabilities of such trunsitions have been developed [4, 5] (see Refs. [6, 7] for a review). We will study the instanton contribution to the total cross section $\sigma_2(E)$ of a process ($2 \to$ any) with two initial particles of the total energy E in the $(-\lambda\phi^4)$-theory. There is a number of arguments showing that $\sigma_2(E)$ can be presented in the following exponential form:

$$\sigma_2(E) \sim e^{\frac{1}{\lambda}F(\epsilon)+\mathcal{O}(1)}, \tag{1}$$

where λ is the coupling constant in the model, $\epsilon = E/E_{sph}$ and E_{sph} is the energy of the sphaleron configuration which characterizes the height of the barrier separating the vacua. The leading order approximation of the function $F(\epsilon)$ for small ϵ was studied in Refs. [7, 8]. The next-to-leading term is a propagator correction for it includes contributions from the propagator in the instanton background. Hence, calculation of the next-to-leading correction requires knowledge of the instanton propagator. It turns out that in the $(-\lambda\phi^4)$-theory an exact expression for the instanton propagator can be obtained. Calculation and discussion of the propagator correction to the function $F(\epsilon)$ is one of the purposes of this article.

An important issue is that of the validity of formula (1). In the electroweak theory a proof based on the properties of the propagator in the instanton background was given in Ref.[9]. We apply the arguments of Ref. [9] in the $(-\lambda\phi^4)$-theory making use of the explicit expression for the propagator.

According to arguments of Refs. [10, 11] for the multiparticle initial state the total cross section is semiclassical and has the form

$$\sigma_N(E) \sim e^{\frac{1}{\lambda}F(\epsilon,\nu)+\mathcal{O}(1)}, \tag{2}$$

where N is the number of initial particles, $\nu = N/N_{sph}$, and $N_{sph} \sim 1/\lambda$ is a characteristic number of particles contained in the sphaleron. Note that in the regime $\lambda \to 0$ and ν fixed $N \sim \nu/\lambda$ is a large

number. The function $F(\epsilon,\nu)$ for the $(-\lambda\phi^4)$-theory was calculated numerically for a certain range of ϵ and ν in Ref. [8]. In Refs. [10, 11, 12] it was argued that the leading exponential term of the two-particle cross section can be calculated from the following formula:

$$\lim_{\lambda\to 0}\lambda\ln\sigma_2=\lim_{\nu\to 0}F\left(\frac{E}{E_{sph}},\nu\right). \tag{3}$$

In this conjecture it is assumed that the limit $\nu\to 0$ exists. The problem is that the function $F(E/E_{sph},\nu)$ is known to contain contributions singular in ν. In particular, in the $(-\lambda\phi^4)$-theory such contributions already appear in the propagator correction. The conjecture basically claims that terms singular in ν cancel each other in the final answer. Its validity, of course, means that the semiclassical form of the two-particle cross section is indeed given by Eq. (1) with $F(E/E_{sph})=F(E/E_{sph},0)$. Verification of conjecture (3) in the next-to-leading order is another purpose of this paper. Note that different arguments in favor of this conjecture were given in Refs.[13, 14].

The plan of the article is the following. In Sect. 2 we describe the model and discuss the propagator in the instanton background. Namely, we present the high energy asymptotics of the propagator and discuss the implementation of Mueller's idea in the scalar model. We also discuss the exact expression of the double residue of the instanton propagator. In Sect. 3 we apply it for the evaluation of the next-to-leading order (propagator correction) of the function $F(\epsilon,\nu)$. There we explicitly demonstrate the appearance of terms singular in ν for $\nu\to 0$ and their cancellation in the final result. Sect. 4 contains some discussion of the results. In particular, the range of validity of the next-to-leading order approximation is estimated.

2 Instanton propagator in the scalar model

We consider the model of one component real scalar field, defined by the Minkowskian action

$$S=\int d^4x\left[\frac{1}{2}\left(\partial_\mu\phi\right)^2-\frac{1}{2}m^2\phi^2+\frac{\lambda}{4!}\phi^4\right], \tag{4}$$

where $\lambda>0$. The potential of the model is unbounded from below, hence the minimum $\phi=0$ is metastable. Underbarrier tunelling from this vacuum to the instability region and its return to the trivial vacuum is the transition which gives rise to the shadow process we are going to study here.

Let us consider first the case $m=0$. There is a well known instanton solution in the massless theory given by the formula [15, 16]

$$\phi_{inst}(x;x_0,\rho)=\frac{4\sqrt{3}}{\sqrt{\lambda}}\frac{\rho}{(x-x_0)^2+\rho^2}. \tag{5}$$

Here $x_{0\mu}$ is the center of the instanton and ρ is its size. Due to the conformal invariance of the massless theory the action of the instanton does not depend on ρ,

$$S^{(0)}_{inst}\equiv S(\phi_{inst})=\frac{16\pi^2}{\lambda}. \tag{6}$$

In the case $m\neq 0$ the mass term breaks the conformal invariance. Using standard scaling arguments it can be shown that there are no regular solutions of the Euclidean equations of motion with finite action. The decay of the vacuum $\phi=0$ is dominated by the constrained instanton, a configuration which can be regarded as an approximate solution of the equations of motion. It minimizes the action under the constraint that the size of the configuration is ρ. A formalism for construction of such configurations and evaluation of the functional integral was developed in [17].

When $\rho m\ll 1$ the constrained instanton configuration behaves like the instanton (5) of the massless theory at $x\ll\rho$ and as a solution of the free massive theory for $x>m^{-1}$. The action of such configuration is

$$S_{inst}(\rho)=\frac{16\pi^2}{\lambda}-\frac{24\pi^2}{\lambda}(\rho m)^2\left[\ln\frac{\rho^2m^2}{4}+2C_E+1\right]+\mathcal{O}(\rho^4m^4), \tag{7}$$

where $C_E=0.577\ldots$ is the Euler constant. For the class of constraints mentioned above the terms given in (7) do not depend on the explicit form of the constraint, whereas the corection $\mathcal{O}(\rho^4m^4)$ does. In our analysis we limit ourselves to the constraint independent order of the approximation.

For $m \neq 0$ the potential barrier separating the trivial vacuum $\phi = 0$ from the instability region is finite. Its height is characterized by a sphaleron solution, a static $SO(3)$-symmetric configuration satisfying the equation of motion. In Ref. [8] it was found that the sphaleron energy and the sphaleron number of particles are

$$E_{sph} = \kappa \frac{m}{\lambda}, \quad \kappa = 113.4 \quad \text{and} \quad N_{sph} = 63 \frac{1}{\lambda} \tag{8}$$

respectively.

Now let us study the propagator in the instanton background. It is defined by the operator $\hat{D}_x$ of quadratic fluctuations appearing in the expansion of the action around the instanton solution. In the massless case this operator is equal to

$$\hat{D}_x = -\frac{\partial^2}{\partial x_\mu^2} + \frac{\lambda}{2}\phi_{inst}^2(x;0,\rho) = -\frac{\partial^2}{\partial x_\mu^2} + \frac{24\rho^2}{(\rho^2+x^2)^2}.$$

It can be easily seen that it possesses five zero modes $\psi_A(x)$ $(A = 1,2,3,4,5)$ corresponding to the translational invariance and the scale invariance of the massless theory. The zero modes can be obtained by differentiation of the instanton solution with respect to the parameters x_0 and ρ:

$$\psi_A \sim \frac{\partial}{\partial \zeta_A}\phi_{inst}(x;x_0,\rho)|_{x_0=0}, \quad \zeta_\mu = (x_0)_\mu, \quad \zeta_5 = \rho. \tag{9}$$

Because of the existence of the zero modes there is an ambiguity in the definition of the propagator that can be fixed by imposing additional constraints. Let $G_f(x,y)$ be the inverse of $\hat{D}_x$ on the subspace of functions orthogonal to some functions $f_A(x)$. The latter satisfy the only condition that the matrix

$$\Omega_{AB} = \int dx \psi_A(x) f_B(x)$$

is invertible [18]. According to this definition the instanton propagator satisfies the equation

$$\hat{D}_x G_f(x,y) = \delta(x-y) - \sum_A f_A(x)\Omega_{AB}^{-1}\psi_B(x) \tag{10}$$

and the orthogonality constraints

$$\int dx f_A(x) G_f(x,y) = 0 = \int dy G_f(x,y) f_B(y). \tag{11}$$

The r.h.s. of Eq. (10) is the projector onto the subspace orthogonal to the functions $f_A(x)$. Physical results, of course, do not depend on particular choice of the functions $f_A(x)$. Below, following the ideas of Refs. [19, 9], we will use the freedom of choosing constraints (11) to eliminate the leading asymptotics of the instanton propagator and simplify the analysis of the initial-state corrections.

For a particularly simple and natural choice of the functions $f_A(x)$, namely for

$$f_A(x) = w(x)\psi_A(x), \tag{12}$$

where the weight function

$$w(x) = \frac{4\rho^2}{(\rho^2+x^2)^2},$$

the propagator in the instanton background was calculated explicitly in Ref. [20] (see also [21]). Allowing some abuse of notation we denote this propagator by $G_\psi(x,y)$. It is equal to

$$\begin{aligned} G_\psi(x,y) &= \frac{1}{2\pi^2}\frac{\rho^2}{(\rho^2+x^2)(\rho^2+y^2)}\left\{\frac{1}{2d(x,y)} - 3\ln d(x,y)\right. \\ &- \left.\frac{43}{5} + 6d(x,y)\ln d(x,y) + \frac{56}{5}d(x,y)\right\}, \end{aligned} \tag{13}$$

where

$$d(x,y) = \frac{\rho^2 (x-y)^2}{(\rho^2+x^2)(\rho^2+y^2)}.$$

The relation between $G_\psi(x,y)$ and the propagator $G_f(x,y)$ for an arbitrary constraint (11) is given by the following formula:

$$\begin{aligned} G_f(x,y) &= G_\psi(x,y) - \left(\int dz G_\psi(x,z) f_A(z) \right) \Omega^{-1}_{AB} \psi_B(y) \\ &- \psi_A(x) \left(\Omega^T_{AB} \right)^{-1} \int dz f_B(z) G\psi(z,y) \\ &+ \psi_A(x) \left(\Omega^T_{AB} \right)^{-1} \left(\int dz dz' f_B(z) G\psi(z,z') f_C(z') \right) \Omega^{-1}_{CD} \psi_D(y). \end{aligned} \tag{14}$$

The Fourier transform of the instanton propagator is defined in the standard way:

$$G_f(p,q) = \int dx dy e^{ipx+iqy} G_f(x,y).$$

In principle, using the exact result, Eq. (13), the function $G_\psi(p,q)$ can be obtained by direct calculation. We did not find the complete expression. Instead we derived the asymptotic formula for the Fourier transform of the instanton propagator in the regime when p^2, q^2 are fixed and $s \equiv (p+q)^2 \to \infty$. The growing terms of the asymptotics are given by

$$G_\psi(p,q) = \frac{16\pi^2}{p^2 q^2} \left[s\rho^2 \ln(s\rho^2) \Pi_1(p,q) + (s\rho^2) \Pi_2(p,q) + \ln(s\rho^2) \Pi_3(p,q) + \ldots \right], \tag{15}$$

where

$$\Pi_1(p,q) = \frac{3}{4} \mathcal{S}_1(p\rho) \mathcal{S}_1(q\rho), \tag{16}$$

$$\Pi_2(p,q) = \frac{3}{2} \left(C_E - \frac{1}{15} - \ln 2 \right) \mathcal{S}_1(p\rho) \mathcal{S}_1(q\rho), \tag{17}$$

$$\begin{aligned} \Pi_3(p,q) &= \left\{ \mathcal{S}_1(p\rho) \left[\frac{9}{2} \mathcal{S}_2(q\rho) - \left(\frac{27}{4} + \frac{3}{4} q^2\rho^2 \right) \mathcal{S}_1(q\rho) \right] \right. \\ &+ \left. \left[\frac{9}{2} \mathcal{S}_2(p\rho) - \left(\frac{27}{4} + \frac{3}{4} p^2\rho^2 \right) \mathcal{S}_1(p\rho) \right] \mathcal{S}_1(q\rho) - \frac{3}{2} \mathcal{S}_2(p\rho) \mathcal{S}_2(q\rho) \right\}. \end{aligned} \tag{18}$$

Here $\mathcal{S}_n(z)$ is defined by $\mathcal{S}_n(z) = z^n K_n(z)$, where $K_n(z)$ is the modified Bessel function. Using the explicit expressions for the translational zero modes (see Eq. (9)) normalized with respect to the weight function $w(x)$, the first two terms of the asymptotic expansion (15) can be written as

$$G_\psi(p,q) = -\frac{1}{5\rho^2} \ln(\rho^2 s) \psi_\mu(p) \psi_\mu(q) - \frac{2}{5\rho^2} \left(C_E - \frac{1}{15} - \ln 2 \right) \psi_\mu(p) \psi_\mu(q) + \ldots \tag{19}$$

The leading term of the asymptotics of the propagator in the instanton background was calculated in Ref. [22]. This result is in complete agreement with the first term in Eq. (19).

In Ref. [9] Mueller proposed an idea to use the ambiguity in the choice of the function $f_A(x)$ in order to cancel the two leading terms in the asymptotics of the propagator $G_\psi(p,q)$. Then the propagator contribution, as well as loop contributions of the initial state corrections disappear. As a consequence, such corrections do not exponentiate, i.e., do not give contributions to the function $F(\epsilon)$. In addition, in this case the initial-final state corrections can be described semiclassically. Namely, the effect of the initial state lines can be taken into account by substituting the instanton by a new field configuration which is a solution to the classical equation of motion with an external source (see Ref. [9] for details).

Now we explain how the functions $f_A(x)$ can be chosen to provide vanishing of the two leading terms of the asymptotics of $G_f(p,q)$. For this we repeat the arguments of Ref. [9]. It turns out that for such functions the corresponding propagator constraint (11) is not relativistically covariant. Let p_1 and p_2

be the arguments of the Fourier transform of the propagator. We choose a coordinate system such that $p_{1j} = p_{2j} = 0$ for $j = 2, 3$, whereas $p_{1+}\rho = p_2\rho \gg 1$ and p_1^2 and p_2^2 are fixed. Here the $\pm$ components of the momenta are defined by

$$p_{j\pm} = \frac{(p_j)_0 \pm (p_j)_1}{\sqrt{2}}.$$

Then $(p_1, p_2)\rho^2 \approx p_{1+}p_{2-}\rho^2 \gg 1$. Only the components $f_\mu(x)$, corresponding to translations, modify the leading asymptotics of the propagator. The Fourier transforms $\tilde{f}_\mu(p)$ of the functions $f_\mu(x)$, defining the required propagator constraint, are chosen in the following way:

$$\tilde{f}_\mu(p) = \delta(p_+ - M)\delta(p_- + M)\bar{f}_\mu(p_2, p_3), \tag{20}$$

where M is an arbitrary parameter of the dimension of mass. Substituting these functions into Eq. (14) one finds after some calculations that

$$G_f(p_1, p_2) = \psi_\mu(p_1)\mathcal{G}_{\mu\nu}\psi_\nu(p_2), \tag{21}$$

where the 4×4 constant matrix $\mathcal{G}_{\mu\nu}$ is equal to

$$\mathcal{G}_{\mu\nu} = -\frac{1}{5\rho^2}\Omega^{-1}_{\mu\sigma}\frac{1}{(2\pi)^8}\int d^4q_1 d^4q_2 f_\sigma(-q_1)\psi_\rho(q_1)\ln\frac{(q_1, q_2)}{M^2}\psi_\rho(q_2)f_\tau(-q_2)\left(\Omega^T\right)^{-1}_{\tau\nu}.$$

Using the freedom of choosing the functions $\bar{f}_\mu$ one can make the constant real symmetric matrix $\mathcal{G}_{\mu\nu}$ equal to zero. We would like to stress that the knowledge of the exact formulas for the leading terms of the asymptotics of $G_\psi(p, q)$, Eqs. (15) - (19), allows us to get the explicit expression of the matrix $\mathcal{G}_{\mu\nu}$. This is in contrast with the case of the electroweak theory where only a general structure of the analogous matrix can be derived [9].

For the perturbative calculations of the function $F(\epsilon, \nu)$ the on-mass-shell residue of the instanton solution will be needed. By definition it is equal to

$$R_{inst}(\mathbf{k}) = (k^2 + m^2)\tilde{\phi}_{inst}(k; 0, \rho)\Big|_{k_0 = i\omega_\mathbf{k}}, \tag{22}$$

where $\tilde{\phi}_{inst}(k; x_0, \rho)$ is the Fourier transform of the instanton,

$$\tilde{\phi}_{inst}(k; x_0, \rho) = \int d^4x e^{ikx}\phi_{inst}(x; x_0, \rho)$$

and $\omega_\mathbf{k} = \sqrt{\mathbf{k}^2 + m^2}$. For the instanton solution (5) in the massless theory

$$R_{inst} = \frac{1}{\sqrt{\lambda}}16\sqrt{3}\pi^2\rho. \tag{23}$$

Correspondingly, to calculate of the next-to-leading correction to the function $F(\epsilon, \nu)$ we need the expressions for the double on-mass-shell residues of the instanton propagator. We will consider the propagator orthogonal to functions (12). Let us introduce the following notations:

$$R_{aa}(\mathbf{k}, \mathbf{q}) = (k^2 + m^2)(q^2 + m^2)\, G_\psi(k_0, \mathbf{k}; q_0, \mathbf{q})|_{k_0 = i\omega_\mathbf{k}, q_0 = i\omega_\mathbf{q}}, \tag{24}$$

$$R_{ab}(\mathbf{k}, \mathbf{q}) = (k^2 + m^2)(q^2 + m^2)\, G_\psi(k_0, \mathbf{k}; q_0, -\mathbf{q})|_{k_0 = i\omega_\mathbf{k}, q_0 = -i\omega_\mathbf{q}}, \tag{25}$$

$$R_{bb}(\mathbf{k}, \mathbf{q}) = (k^2 + m^2)(q^2 + m^2)\, G_\psi(k_0, -\mathbf{k}; q_0, -\mathbf{q})|_{k_0 = -i\omega_\mathbf{k}, q_0 = -i\omega_\mathbf{q}}, \tag{26}$$

The indices a and b correspond to initial and final particles, respectively (in the notations of Ref. [11]). For the scalar massless field $\omega_\mathbf{k} = |\mathbf{k}|$ and all three residues (24)-(26) can be expressed in terms of one function:

$$R_\#(\mathbf{k}, \mathbf{q}) = \rho^2 R\left(\rho^2 s^{(0)}_\#(\mathbf{k}, \mathbf{q})\right), \tag{27}$$

where $\# = aa$, ab, bb and the function $s^{(0)}_\#(\mathbf{k}, \mathbf{q})$ is the s-variable for the corresponding particles on the mass shell,

$$s^{(0)}_{aa}(\mathbf{k}, \mathbf{q}) = s^{(0)}_{bb}(\mathbf{k}, \mathbf{q}) = -s^{(0)}_{ab}(\mathbf{k}, \mathbf{q}) = -2(|\mathbf{k}||\mathbf{q}| - \mathbf{kq}).$$

However, in the calculation of the next-to-leading order corrections due to non-zero mass must be taken into account. It turns out that within the accuracy set by Eq. (7) it is enough to consider the residues defined through the relation

$$R_{\#}(\mathbf{k},\mathbf{q}) = \rho^2 R\left(\rho^2 s_{\#}(\mathbf{k},\mathbf{q})\right), \tag{28}$$

(cf. (27)), where the function is calculated for the instanton propagator of the *massless* theory, whereas the s-variable is taken for the *massive* one:

$$\begin{aligned} s_{aa}(\mathbf{k},\mathbf{q}) &= s_{bb}(\mathbf{k},\mathbf{q}) = -2m^2 - 2(\omega_{\mathbf{k}}\omega_{\mathbf{q}} - \mathbf{kq}), & (29)\\ s_{ab}(\mathbf{k},\mathbf{q}) &= -2m^2 + 2(\omega_{\mathbf{k}}\omega_{\mathbf{q}} - \mathbf{kq}). & (30) \end{aligned}$$

The consistency of this procedure is discussed in Sect. 4.

The *exact* expression for the function $R(s)$ was obtained in Ref. [21] and is given by

$$\begin{aligned} R(s) &= 16\pi^2\left\{\alpha_1\left[s\ln\frac{s}{4} + 2\left(C_E - \frac{1}{15}\right)s\right] - \alpha_2\left[\ln\frac{s}{4} + 2\left(C_E + \frac{43}{30}\right)\right]\right\}, & (31)\\ \alpha_1 &= 3/4, \quad \alpha_2 = 3/2. \end{aligned}$$

In the next section this result will be used for the calculation of the next-to-leading correction to the function $F(\epsilon,\nu)$.

3 Multiparticle cross section

Formula (2) for the multiparticle cross-section of shadow processes comes from the following expression derived in Ref. [11],

$$\begin{aligned} \sigma_N(E) &\sim \int d^4x_0 d\rho d^4\xi d\theta \exp\left[-2S_{inst}(\rho) + \frac{1}{\lambda}W^{(1)}(x_0,\rho,\xi,\theta)\right.\\ &+ \left.\frac{1}{\lambda}W^{(2)}(x_0,\rho,\xi,\theta) + \ldots\right], \end{aligned} \tag{32}$$

where we integrate over the position x_0 and the size ρ of the instanton, as well as over auxiliary variables ξ_μ and θ. We also indicated explicitly the dependence of the action on the size of the instanton (see Eq. (7)). The terms $W^{(i)}$ account for fluctuations in the instanton background: $W^{(1)}$ corresponds to leading diagrams without propagator lines, $W^{(2)}$ corresponds to diagrams with one internal propagator in the instanton background, etc. Diagrams with loops do not appear in the $(1/\lambda)$ order of the semi-classical approximation, they contribute to $\mathcal{O}(1)$ terms in (2).

General expressions for the functions $W^{(1)}$ and $W^{(2)}$ were derived in [11]. The integrals in Eq. (32) are evaluated by the saddle point method. It can be checked that up to the next-to-leading order the saddle point values of x_0, ρ, ξ and θ are determined by the leading-order equations. These equations are obtained by differentiation of the expression $(-2S_{inst}(\rho) + W^{(1)}/\lambda)$ with respect to x_0, ρ, ξ and θ. The physically relevant saddle point has $(x_0)_i = 0$, $\xi_i = 0$ $(i = 1,2,3)$, while $(x_0)_0$, ξ_0 and θ are purely imaginary. It is convenient to introduce the following notations: $x_0 = i\tau$, $\xi_0 = i\chi$, and $\theta = -i\ln\gamma$. In accordance with Eq. (32) the function $F(\epsilon,\nu)$ is represented as

$$F(\epsilon,\nu) = -32\pi^2 + F^{(1)}(\epsilon,\nu) + F^{(2)}(\epsilon,\nu) + \ldots. \tag{33}$$

The first term in the r.h.s. is just $(-2\lambda S^{(0)}_{inst})$, where $S^{(0)}_{inst}$ is the instanton action in the massless theory, Eq. (6). The non-trivial leading order correction $F^{(1)}(\epsilon,\nu)$ corresponds to the contribution of

$$2\lambda(S^{(0)}_{inst} - S_{inst}(\rho)) + W^{(1)}$$

in Eq. (32). The next-to-leading (propagator) correction $F^{(2)}(\epsilon,\nu)$ is given by $W^{(2)}$ evaluated at the saddle point solution.

3.1 Leading order correction

For general values of ϵ and ν the system of saddle point equations is too complicated and we studied it numerically. The results are described at the end of the section.

In the limit of small ν the calculations simplify considerably. Keeping only relevant terms we obtain that the function $W^{(1)}$ of Eq. (32) reads

$$\begin{aligned}\frac{1}{\lambda}W^{(1)}(\tau,\rho,\chi,\gamma) &= E\chi - N\ln\gamma + \frac{1}{(2\pi)^3}\int\frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}R_{inst}(\mathbf{p})e^{-\omega_{\mathbf{p}}\tau}R_{inst}(\mathbf{p}) \\ &+\frac{\gamma}{(2\pi)^3}\int\frac{d\mathbf{p}}{2\omega_{\mathbf{p}}}R_{inst}(\mathbf{p})e^{-\omega_{\mathbf{p}}(\chi-\tau)}R_{inst}(\mathbf{p}) + \dots \\ &= E\chi - N\ln\gamma + 192\pi^2\frac{\rho^2m^2}{\lambda}\left[\Phi(m\tau)+\frac{\gamma}{m^2(\chi-\tau)^2}\right]+\dots,\end{aligned} \tag{34}$$

where R_{inst} is given by expression (23), $\omega_{\mathbf{p}}=\sqrt{\mathbf{p}^2+m^2}$, and

$$\Phi(z)\equiv\frac{1}{z}K_1(z).$$

We would like to stress that in these calculations the expression for the energy $\omega_{\mathbf{p}}$ of the massive theory is used, whereas it is enough to substitute the residue R_{inst} of the instanton solution of the massless theory. This is consistent with the approximation we are considering in the present paper. The question of validity of this procedure is discussed in Sect. 4.

For further calculations it is convenient to introduce the variables $\tilde\epsilon = E\lambda/m$ and $\tilde\nu = N\lambda$. From Eqs. (8) it follows that $\tilde\epsilon=\kappa\epsilon$ and $\tilde\nu = 63\nu$. To the leading order in ν the saddle point solution can be written in the form

$$\tilde\rho^2 = -\frac{1}{192\pi^2m^2}\frac{\tilde\epsilon}{\Phi'(m\tilde\tau)};\quad \tilde\gamma=-4\left(\frac{\tilde\nu}{\tilde\epsilon}\right)^3\Phi'(m\tilde\tau);\quad \tilde\chi=\tilde\tau+\frac{2}{m}\frac{\tilde\nu}{\tilde\epsilon}. \tag{35}$$

Here the prime denotes the derivative, $\ln C = -\ln 4 + 2C_E + 1$, and $\tilde\tau=\tilde\tau(\epsilon)$ is determined by the equation

$$\ln\left(-\frac{\tilde\epsilon Ce}{192\pi^2\Phi'(m\tilde\tau)}\right)+4\Phi(m\tilde\tau)=0. \tag{36}$$

Note that the saddle point solution satisfies the relation

$$\frac{2\tilde\gamma}{m^3(\tilde\chi-\tilde\tau)^3}+\Phi'(m\tilde\tau)=0, \tag{37}$$

which will be used later.

Substituting the saddle point solution into Eq. (34) we obtain the leading order contribution $F^{(1)}$:

$$F^{(1)}(\epsilon,\nu)=\kappa\epsilon\left[m\tilde\tau_0(\epsilon)+\frac{1}{4\Phi'(m\tilde\tau_0(\epsilon))}\right]+\mathcal{O}(\nu). \tag{38}$$

In the limit $\epsilon\to 0$ Eq. (36) can be solved iteratively. One gets

$$m\tilde\tau(\epsilon)=\frac{2}{\sqrt{\ln\frac{1}{\epsilon}}}+\frac{\ln\ln\frac{1}{\epsilon}}{\left(\ln\frac{1}{\epsilon}\right)^{3/2}}+\dots \tag{39}$$

Then in the leading order in energy solutions (35) become

$$(m\tilde\rho)^2=\frac{1}{48\pi^2}\frac{\tilde\epsilon}{\left(\ln\frac{1}{\tilde\epsilon}\right)^{3/2}};\quad \tilde\gamma=\left(\frac{\tilde\nu}{\tilde\epsilon}\right)^3\left(\ln\frac{1}{\tilde\epsilon}\right)^{3/2};\quad m(\tilde\chi-\tilde\tau)=2\frac{\tilde\nu}{\tilde\epsilon}. \tag{40}$$

In this regime the function $F^{(1)}(\epsilon,\nu)$ is equal to

$$F^{(1)}(\epsilon,\nu)=2\frac{\kappa\epsilon}{\sqrt{\ln\frac{1}{\epsilon}}}\left[1+\mathcal{O}\left(\frac{\ln\ln\frac{1}{\epsilon}}{\ln\frac{1}{\epsilon}}\right)\right]+\mathcal{O}(\nu). \tag{41}$$

3.2 Propagator correction

The next-to-leading order function $W^{(2)}$ can be written as the sum of contributions involving the propagator between final states, between initial and final states and between initial states, respectively:

$$W^{(2)} = W^{(2)}_{(f-f)} + W^{(2)}_{(i-f)} + W^{(2)}_{(i-i)}. \tag{42}$$

As we have already mentioned the expressions for these terms are given in Ref. [11]. The complete propagator correction was calculated numerically, the results are discussed in Sect. 3.3. Here we study the propagator correction analytically in the limit of small ν. Keeping only relevant contributions we obtain that

$$\frac{1}{\lambda}W^{(2)}_{(f-f)} = I_{bb}(\tau,\tau) + \ldots, \tag{43}$$

$$\frac{1}{\lambda}W^{(2)}_{(i-f)} = 2\gamma I_{ab}(\tau,\chi-\tau) + \ldots, \tag{44}$$

$$\frac{1}{\lambda}W^{(2)}_{(i-i)} = \gamma^2 I_{aa}(\chi-\tau,\chi-\tau) + \ldots, \tag{45}$$

where

$$\begin{aligned} I_{\#}(\tau_1,\tau_2) &= \frac{1}{(2\pi)^6 m^2}\int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}}\frac{d\mathbf{q}}{2\omega_{\mathbf{q}}} R_{inst}(\mathbf{k}) e^{-\omega_{\mathbf{k}}\tau_1} R_{\#}(\mathbf{k},\mathbf{q}) R_{inst}(\mathbf{q}) e^{-\omega_{\mathbf{q}}\tau_2} \\ &= 48\frac{\rho^4}{\lambda}\int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}}\frac{d\mathbf{q}}{2\omega_{\mathbf{q}}} e^{-\omega_{\mathbf{k}}\tau_1}\frac{R\left(\rho^2 s_{\#}(\mathbf{k},\mathbf{q})\right)}{16\pi^2} e^{-\omega_{\mathbf{q}}\tau_2}. \end{aligned} \tag{46}$$

The functions $R_{\#}(\mathbf{k},\mathbf{q})$ and $R(\rho^2 s)$ are given by Eqs. (28) - (31), all necessary notations were introduced in Sect. 2.

In the limit of small ν the expression for the propagator correction in terms of simple integrals can be obtained. However, it is quite cumbersome and we do not present this result here. Instead we calculate and analyze groups of terms which are singular in ν. From Eqs. (35) it follows that for the saddle point solution in the limit $\nu \to 0$ we have

$$m(\tilde{\chi}-\tilde{\tau}) \sim \nu \to 0, \quad \tilde{\gamma} \sim \nu^3. \tag{47}$$

Using these properties it is easy to select and calculate the terms in Eqs. (44) and (45) which are singular in ν. Evaluating these terms at the saddle point solution (35), we obtain that

$$\begin{aligned} F^{(2)}_{(i-i)} &= -32(192\pi^2)\alpha_1\tilde{\rho}^6\frac{\tilde{\gamma}^2}{(\tilde{\chi}-\tilde{\tau})^6}\left[\ln\frac{\tilde{\rho}^2}{(\tilde{\chi}-\tilde{\tau})^2}+\ldots\right] \\ &= \frac{8\alpha_1}{(192\pi)^2}\frac{\tilde{\epsilon}^3}{\Phi'(m\tilde{\tau})}\left[2\ln\frac{1}{\tilde{\nu}} + \ln\left(-\frac{\tilde{\epsilon}^3}{768\pi^2\Phi'(m\tilde{\tau})}\right)+\ldots\right], \end{aligned} \tag{48}$$

$$\begin{aligned} F^{(2)}_{(i-f)} &= 16(192\pi^2)\alpha_1\tilde{\rho}^6\frac{2\tilde{\gamma}}{(\tilde{\chi}-\tilde{\tau})^3}\left[-\Phi'(m\tilde{\tau}_0)\ln\frac{\tilde{\rho}^2}{\tilde{\tau}(\tilde{\chi}-\tilde{\tau})}+\ldots\right] \\ &= -\frac{16\alpha_1}{(192\pi)^2}\frac{\tilde{\epsilon}^3}{\Phi'(m\tilde{\tau})}\left[\ln\frac{1}{\tilde{\nu}} + \ln\left(-\frac{\tilde{\epsilon}^2}{384\pi^2 m\tilde{\tau}\Phi'(m\tilde{\tau})}\right)+\ldots\right], \end{aligned} \tag{49}$$

where the dots stand for non-singular terms. Summing contributions (48) and (49) one gets

$$\begin{aligned} F^{(2)}_{(i-i)} &+ F^{(2)}_{(i-f)} \\ &= -16(192\pi^2)\alpha_1\tilde{\rho}^6\left[\frac{\tilde{\gamma}}{(\tilde{\chi}-\tilde{\tau})^3}\left(\frac{2\tilde{\gamma}}{(\tilde{\chi}-\tilde{\tau})^3}+\Phi'(m\tilde{\tau})\right)\ln\frac{\tilde{\rho}^2}{(\tilde{\chi}-\tilde{\tau})^2}\right. \\ &+ \left.\frac{\tilde{\gamma}}{(\tilde{\chi}-\tilde{\tau})^3}\Phi'(m\tilde{\tau})\ln\frac{\tilde{\rho}^2}{\tilde{\tau}^2}+\ldots\right] \end{aligned} \tag{50}$$

$$= -\frac{8\alpha_1}{(192\pi)^2}\frac{\tilde{\epsilon}^3}{\Phi'(m\tilde{\tau})}\left[\ln\left(-\frac{\tilde{\epsilon}}{192\pi^2 m\tilde{\tau}\Phi'(m\tilde{\tau})}\right)+\ldots\right]. \tag{51}$$

As we see from the last line, Eq. (51), the singular terms $\ln(1/\nu)$ cancel each other. Eq. (50), reveals the reason of this cancellation: due to relation (37) the coefficient of the term $\ln(\tilde{\rho}^2/(\tilde{\chi}-\tilde{\tau})^2)$, which gives rise to the singularity $\ln(1/\nu)$, is equal to zero exactly. This result is general and does not depend on any approximation.

We would like to remark that the terms singular in ν are proportional to α_1. From Eq. (31) it follows that they originate from the terms proportional to $s\ln s$ and s in the residue of the instanton propagator. If one uses the instanton propagator $G_f(p,q)$ satisfying constraint (11) with the functions f_A such that two leading terms in the asymptotics (15) vanish, then the leading asymptotics $s\ln s$ and s of the propagator for large s are absent. As a consequence, the singular terms $\ln(1/\nu)$ do not appear.

For energies small enough, such that $m\tilde{\tau} \ll 1$, the expressions simplify further and the result for the next-to-leading correction can be written in a simple form. We obtain that

$$F^{(2)}(\epsilon,\nu) = -\frac{\alpha_2}{192\pi^2}(\tilde{\epsilon}m\tilde{\tau})^2\left[\ln\frac{\tilde{\epsilon}m\tilde{\tau}}{384\pi^2}+\frac{58}{15}+\mathcal{O}(m^2\tilde{\tau}^2)\right]. \tag{52}$$

In the limit $\epsilon \to 0$ we use solution (39) and obtain

$$F^{(2)}(\epsilon,\nu) = \frac{4\alpha_2\kappa^2\epsilon^2}{192\pi^2}\left(1+\frac{1}{2}\frac{\ln\ln\frac{1}{\epsilon}}{\ln\frac{1}{\epsilon}}+\dots\right) = \frac{\kappa^2\epsilon^2}{32\pi^2}\left(1+\frac{1}{2}\frac{\ln\ln\frac{1}{\epsilon}}{\ln\frac{1}{\epsilon}}+\dots\right) \tag{53}$$

We see that at low energies the main contribution is proportional to α_2, i.e. comes from the $\ln s$ and constant terms in the residue (31) of the instanton propagator. In fact it is easy to check that it is precisely the term $\sim \ln s$ in Eq. (31) which gives the contribution (53).

3.3 Numerical results

For arbitrary ϵ and ν the functions $F^{(1)}(\epsilon,\nu)$ and $F^{(2)}(\epsilon,\nu)$ were studied numerically. It turned out that the saddle point solution exists only for a certain region in the (ϵ,ν)-plane. It lies inside the rectangle $0<\epsilon<\epsilon_{max}=0.55$ and $0<\nu<\nu_{max}=0.25$. We performed the numerical analysis for this whole region.

To present the results it is convenient to introduce the following functions:

$$\mathcal{F}_1(\epsilon,\nu) = 1-\frac{F^{(1)}(\epsilon,\nu)}{32\pi^2}, \quad \mathcal{F}_2(\epsilon,\nu) = 1-\frac{F^{(1)}(\epsilon,\nu)+F^{(2)}(\epsilon,\nu)}{32\pi^2}.$$

They are normalized by the conditions $\mathcal{F}_1(0,\nu)=\mathcal{F}_2(0,\nu)=1$.

Lines of constant $\mathcal{F}_2(\epsilon,\nu)$ are plotted in Fig. 1. We want to study the cross section for shadow processes with a few initial particles. Then according to conjecture (3) points where the lines cross the $\nu=0$ axis are of particular interest. For example, $\mathcal{F}_2(\epsilon,0)=0.95$ at $\epsilon=0.180$, $\mathcal{F}_2(\epsilon,0)=0.85$ at $\epsilon=0.492$. We would like to mention that, in fact, in the studied region of (ϵ,ν) the propagator correction is quite small comparing to the leading order. Thus, the difference between $\mathcal{F}_1(\epsilon,\nu)$ and $\mathcal{F}_2(\epsilon,\nu)$ does not exceed 10^{-2}.

The curves in Fig. 1 end at the line formed by saddle points corresponding to the periodic instanton solutions. For them $\tilde{\tau}(\epsilon,\nu)=\tilde{\chi}(\epsilon,\nu)/2$. This line is directed from the zero energy instanton ($\epsilon=\nu=0$) to the sphaleron ($\epsilon=\nu=1$).

As it has been already mentioned, the complete function $F(\epsilon,\nu)$ was calculated numerically in the range $0.4<\epsilon<3.5$ and $0.25<\nu<1$ in Ref. [8]. The computation was performed by solving a certain classical boundary value problem on the lattice. With the size of the lattice used in the numerical calculation in Ref. [8], the authors did not obtain data for smaller ϵ and ν except for the line of the periodic instantons. The comparison shows that our perturbative results do not differ significantly from the exact ones of Ref. [8] for $\epsilon<0.25$ and $\nu<0.2$. These values can be regarded as a rough estimate of the range of validity of the leading and next-to-leading approximations.

4 Discussion and conclusions

In the present paper we have analyzed the multiparticle cross section of the shadow processes induced by instanton transitions in the simple scalar model (4). Using the exact analytical expression for the

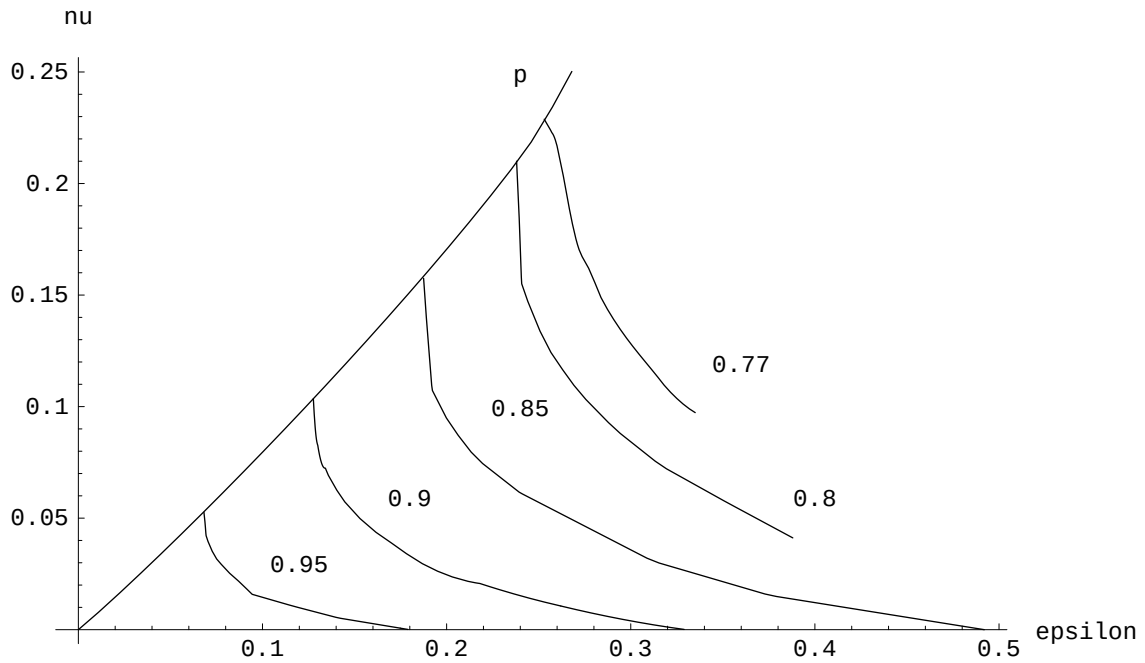

Figure 1: Lines of constant $\mathcal{F}_2(\epsilon,\nu)$ in the (ϵ,ν)-plane. Numbers near the lines indicate the value of $\mathcal{F}_2$, "p" labels the line of periodic instanton solutions.

on-shell residue of the propagator of quantum fluctuations in the instanton background we calculated the suppression factor in the next-to-leading order.

The calculation of the leading and next-to-leading orders of $F(\epsilon,\nu)$ was performed assuming that the size of the instanton solution is small enough, namely $(\tilde{\rho}m) \ll 1$. Neglecting $\mathcal{O}(\rho^4 m^4)$ terms in the action (7) and using the instanton and the residue of the instanton propagator of the *massless* theory in Eqs. (22) and (28) amount to omission of corrections of the type

$$\frac{\rho^2}{\tau^2}(\rho^2 m^2) \tag{54}$$

in $F^{(1)}$ and

$$(\rho m)^4, \quad \frac{\rho^4}{\tau^4}(\rho^2 m^2), \quad \frac{\rho^4}{\tau^4}(\rho^2 m^2)\ln\frac{\rho^2}{\tau^2} \quad \text{and} \quad \frac{\rho^4}{\tau^4}(m^2\tau^2)^k \tag{55}$$

in $F^{(2)}$. We checked numerically that in the region of ϵ and ν, where the saddle point solution exists, the terms in Eqs.(54), (55) are really small. As an illustration let us consider the case of very small ϵ and use the saddle point solution (39), (40). We obtain that

$$\begin{aligned}
\frac{\tilde{\rho}^2}{\tilde{\tau}^2}(\tilde{\rho}^2 m^2) &\sim \frac{\epsilon^2}{\left(\ln\frac{1}{\epsilon}\right)^{5/2}}, \\
(\tilde{\rho}m)^4 &\sim \frac{\epsilon^2}{\left(\ln\frac{1}{\epsilon}\right)^{3}}, \quad \frac{\tilde{\rho}^4}{\tilde{\tau}^4}(\tilde{\rho}^2 m^2) \sim \frac{\epsilon^3}{\left(\ln\frac{1}{\epsilon}\right)^{5/2}}, \\
\frac{\tilde{\rho}^4}{\tilde{\tau}^4}(\tilde{\rho}^2 m^2)\ln\frac{\tilde{\rho}^2}{\tilde{\tau}^2} &\sim \frac{\epsilon^3}{\left(\ln\frac{1}{\epsilon}\right)^{3/2}}, \quad \frac{\tilde{\rho}^4}{\tilde{\tau}^4}(m^2\tilde{\tau}^2)^k \sim \frac{\epsilon^2}{\left(\ln\frac{1}{\epsilon}\right)^{1+k/2}}.
\end{aligned}$$

All these corrections are subleading compared to the terms retained in the function $F^{(2)}(\epsilon,\nu)$, Eq. (53). Contributions due to non-zero mass amount to corrections in powers of $(m\tilde{\tau})$, where $\tilde{\tau}(\epsilon,\nu)$ is the saddle point solution for τ. In general, these corrections are not small, and all of them were taken into account by using the s-variable and the energy $\omega_{\mathbf{k}}$ of massive particles in Eqs. (28) - (30), (34), (46), etc. Our

numerical analysis shows that the inequality $m\tilde{\tau} < 1$ is satisfied, for example, for $\epsilon < 0.4$ if ν is close to $\nu = 0$ and for $\epsilon < 0.02$ for the periodic instanton solutions. Comparing this to the region in the (ϵ, ν)-plane in Fig. 1, for which we carried out the calculation in this article, one can see that our formalism, accounting for arbitrary $m\tilde{\tau}$, allows to enlarge considerably the range of validity of the next-to-leading approximation.

The range of validity of the next-to-leading order approximation of the function $E(\epsilon, \nu)$ was estimated by comparing our results with numerical computations in Ref. [8] for the values of ϵ and ν for which the latter can be translated to the case of shadow processes, i.e., for periodic instantons. The comparison shows that the perturbative results do not differ significantly from the exact ones for $\epsilon \leq 0.25$ and $\nu \leq 0.2$.

For this range of values of ϵ and ν and away from the line of periodic instantons, methods of Ref. [8] do not allow to obtain exact results. Therefore, at the moment our perturbative calculations are the only ones which give quantitative behaviour of the supression factor in this range.

From Eqs. (39) we see that approximate formulas (41) and (53) are valid as long as

$$\frac{\ln \ln \frac{1}{\epsilon}}{\ln \frac{1}{\epsilon}} \ll 1.$$

For this range of energies we obtained the analytical expressions for the suppression factor and values of the saddle point parameters $\tilde{\rho}$, $\tilde{\chi}$, $\tilde{\tau}$ and $\tilde{\gamma}$. Formula (52) for the propagator correction for small ν is valid when $m\tilde{\tau} \ll 1$. According to the estimate, mentioned above, this condition is satisfied if $\epsilon \ll 0.4$. This can be also verified by analyzing Eq. (36).

We also checked the cancellation of terms singular in the limit $\nu \to 0$ in the propagator correction $F^{(2)}$. As we have explained, this is closely related to the problem of quasiclassical evaluation of contributions of initial states and initial-final states. In the article we also discussed this problem within the approach proposed by Mueller. Namely, we calculated the leading asymptotics of the instanton propagator at large s and showed that it can be cancelled by an appropriate choice of the propagator constraint. According to Ref. [9], with such propagator the problem of semiclassical calculation of contributions due to initial states and initial-final states can be tackled properly.

Acknowledgments

We would like to thank A. Ringwald and V. Rubakov for discussions and valuable comments. Y.K. acknowledges financial support from the Russian Foundation for Basic Research (grant 98-02-16769-a) and grant CERN/P/FIS/1203/98.

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