Deep inelastic scattering and the Lamb shift in muonic hydrogen

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Abstract

The proton structure and proton polarizability corrections to the Lamb shift of electronic hydrogen and muonic hydrogen were evaluated on the basis of modern experimental data on deep inelastic structure functions. Numerical value of proton polarizability contribution to (2P-2S) Lamb shift is equal to 4.4 GHz.

The muonic hydrogen $(\mu^- p^+)$ is a two-particle bound state of a muon and a proton. The energy levels of this system are defined as for electronic hydrogen [1]. By virtue of the fact that the mass ratio of electron and muon $m_e/m_\mu = 4.836332 \cdot 10^{-3}$, some quantum electrodynamical contributions, proton structure corrections and proton polarizability effects increase in importance in the case of muonic hydrogen [2, 3]. For example, the electron vacuum polarization gives the main contribution to the Lamb shift of muonic hydrogen (contrary to ordinary hydrogen), because the Compton wavelength of the electron and the Bohr radius of muonic hydrogen have the same order: $\hbar/m_e c : \hbar^2/\mu e^2 = 0.737386$. The experimental measurement of (2P-2S) Lamb shift in muonic hydrogen makes it possible to determine with more high accuracy the proton charge radius $R_p = \sqrt{< r^2 >}$ [4], which is one of the universal fundamental physical constants. Theoretical investigations of the main contributions to the Lamb shift of muonic hydrogen (μp) were done in [2, 3]. The results of calculation of different order corrections, known at present time may be found in [5]. New six-order vacuum polarization contribution to the Lamb shift of the muonic hydrogen was obtained recently in [6]. In this work we consider the Lamb shift contribution in $(\mu^- p^+)$, connected with the proton polarizability effects. As in the case of muonic hydrogen hyperfine splitting [7, 8], we have used the local quasipotential equation for description of two-particle bound state $(\mu^- p^+)$ [9].

The proton polarizability contribution is determined by the amplitude of two - photon muon - proton interaction, shown on fig.1. Compton forward scattering amplitude of the electron, appearing in the two - photon interaction, may be expressed as a sum of direct and crossed two - photon diagrams:

$$M_{\mu\nu}^{(e)} = \bar{u}(q_1)\left[\gamma_\mu \frac{\hat{p_1} + \hat{k} + m_1}{(p_1+k)^2 - m_1^2 + i\epsilon}\gamma_\nu + \gamma_\nu \frac{\hat{p_1} - \hat{k} + m_1}{(p_1-k)^2 - m_1^2 + i\epsilon}\gamma_\mu\right] u(p_1). \tag{1}$$

Neglecting by the relative motion momenta of the particles in initial and final states $(\vec{p} = \vec{q} = 0)$, we can also parameterize the amplitude of the proton virtual Compton scattering in the following manner [10, 11]:

$$M_{\mu\nu}^{(p)} = \bar{v}(p_2)\Bigg\{\frac{1}{2}C_1\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right) + \frac{1}{2m_2^2}C_2\left(p_{2\mu} - \frac{m_2\nu}{k^2}k_\mu\right)\left(p_{2\nu} - \frac{m_2\nu}{k^2}k_\nu\right) + \tag{2}$$

$$+\frac{1}{2m_2}H_1\left([\gamma_\nu,\hat{k}]p_{2\mu} - [\gamma_\mu,\hat{k}]p_{2nu} + [\gamma_\mu,\gamma_\nu]\right) + \frac{1}{2}H_2\left([\gamma_\nu,\hat{k}]k_\mu - [\gamma_\mu,\hat{k}]k_\nu + [\gamma_\mu,\gamma_\nu]k^2\right)\Bigg\}v(q_2),$$

where $\nu = k_0$ is the energy of the virtual photon. Tensor (2) contains symmetric and antisymmetric parts. Symmetric part, which is determined by the structure functions $C_{1,2}(\nu, k^2)$ fixes the contribution to the Lamb shift. Antisymmetric part of expression (2), defined by the functions $H_{1,2}(\nu, k^2)$, gives the contribution to the hyperfine splitting of S - energy levels. To extract the hyperfine splitting part of the

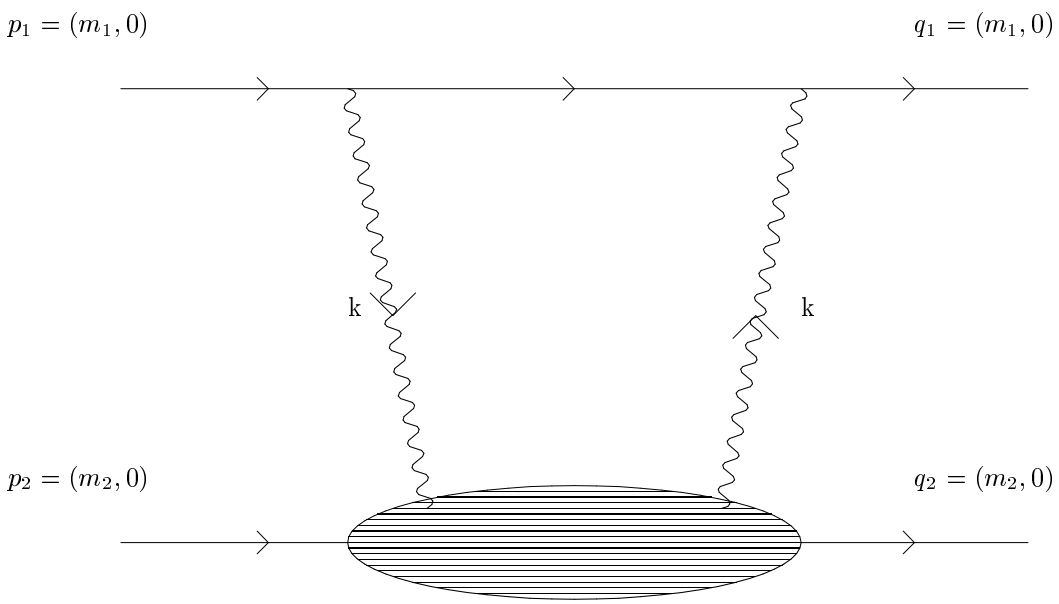

Figure 1: Proton polarizability correction

quasipotential $V^{\rm hfs}$, we can use the projection operators of muon - proton two - particle system on the states with total spin S=0 ($V(^1S_0)$) and S=1 ($V(^3S_1)$):

$$\hat{\Pi} = u(p_1)\bar{v}(p_2) = \frac{1}{2\sqrt{2}}(1+\gamma_0)\hat{\epsilon}(\gamma_5), \tag{3}$$

To construct the S -wave Lamb shift part of the quasipotential it is common practice to use two approaches:

1. The averaging of the amplitudes (1) and (2) over muon and proton spins [12].
2. The use of projection operators (3) and the construction of necessary interaction in the kind:

$$V^{\rm Ls} = V(^1S_0) + \frac{3}{4}V^{\rm hfs}. \tag{4}$$

Multiplying the amplitudes (1) and (2) and extracting the corresponding parts of the quasipotential (hyperfine splitting part and Lamb shift part), we obtain:

$$\left(M^{(\mu)}_{\mu\nu}M^{(p)}_{\mu\nu}\right)^{\rm hfs} = -\frac{16}{3}\frac{k^2}{k^4-4k_0^2m_1^2}\left[m_2(k_0^2+2k^2)H_1 + 3k_0k^2H_2\right], \tag{5}$$

$$\left(M^{(\mu)}_{\mu\nu}M^{(p)}_{\mu\nu}\right)^{\rm Ls} = \frac{2m_1}{k^4-4k_0^2m_1^2}\left[-(2k_0^2+k^2)C_1 + (k^2-k_0^2)C_2\right], \tag{6}$$

Expression (5) was used earlier in the hyperfine splitting calculation [13, 14, 15]. Let consider Lamb shift contribution on the basis of (6). Using dispersion relation for structure functions $C_i(k_0,k^2)$ and explicitly displaying the contribution of the proton intermediate state, shown on fig.2, we have:

$$C_i(k_0,k^2) = C_i^p(k_0,k^2) + \frac{1}{\pi}\int_{\nu_0}^{\infty}\frac{d\nu^2}{(\nu^2-k_0^2)}ImC_i(\nu,k^2), \tag{7}$$

$$\nu_0 = m_\pi + \frac{1}{2m_2}(Q^2+m_\pi^2), \quad Q^2 = -k^2,$$

and what is more the function $C_1(k_0,k^2)$ satisfies to dispersion relation with one subtraction to provide the correct asymptotic on ν of the intgral expression in (7).

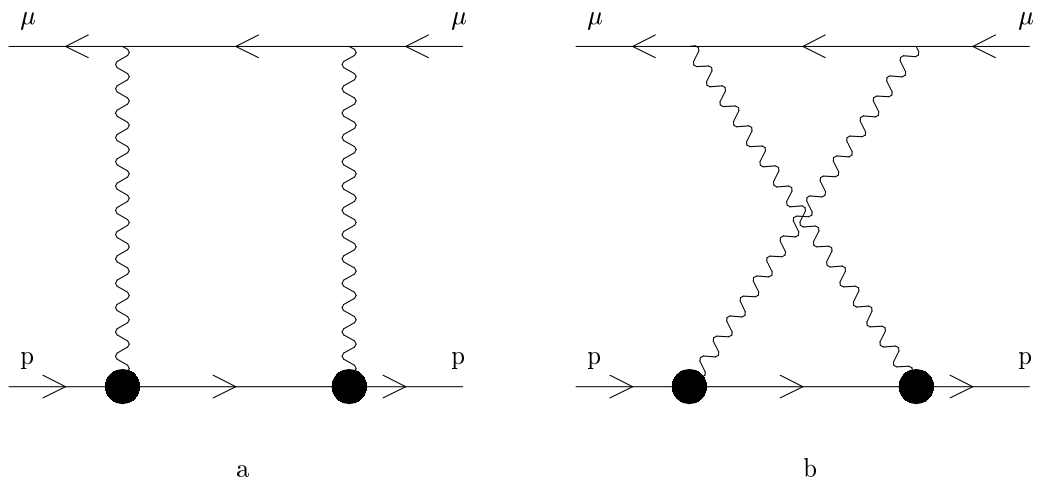

Figure 2: Proton finite - size correction of order $(Z\alpha)^5$.

The proton vertices of two - photon amplitudes are defined by two form factors F_1 and F_2 [16]:

$$\Gamma^{\mu} = \gamma^{\mu} F_1 + \frac{i}{2m_2}\sigma^{\mu\nu}k_{\nu}F_2, \tag{8}$$

The contribution of diagrams (a) and (b) of Fig.2 to the Lamb shift may be expressed in the kind:

$$\Delta E_p^{\rm Ls} = -|\psi(0)|^2\frac{(Z\alpha)^2}{\pi^2}\int d^4k\frac{1}{(k^2)^2}\frac{1}{2m_1}m_2(k^4+4m_1^2k_0^2)\frac{1}{(k^4+4m_2^2k_0^2)} \tag{9}$$

$$\left[4m_2^2(k^4+2k_0^4)F_1^2 - 2k^4(k^2+2k_0^2)F_1F_2 - 3k_0^2k^4F_2^2\right],$$

where we have transformed integration over four - dimensional Euclidian space, changing $k_0 \to ik_0$. It is necessary to say that this integration contains infrared divergence. To remove it we subtract from (8) the correction related with iteration term of the quasipotential and the contribution of two - photon amplitudes in the case of point - like proton. The contribution of iteration part of the quasipotential takes the kind:

$$\Delta E_{\rm iter}^{\rm Ls} = V_{1\gamma}\times G^f\times V_{1\gamma} \approx \frac{\mu^4(Z\alpha)^5}{m_2^2\pi n^3}\int_0^{\infty}\frac{dk}{k^2}\left[F_1'(0)4m_2^2 + F_2(0)\right]. \tag{10}$$

The iteration terms of quasipotential fully eliminate infrared divergences in the hyperfine structure of hydrogen - like atoms [11]. Note that the correction of order $(Z\alpha)^4$ in the Lamb shift of hydrogen atom, depending from proton structure is defined by one - photon term of the quasipotential $V_{1\gamma} \sim G_E(k^2)/\vec{k}^2$ [17] (G_E is the Sachs form factor) in the following form:

$$E^{\rm Ls} = \frac{2\mu^3}{3n^3}(Z\alpha)^4R_p^2\delta_{l0}, \quad R_p^2 = \frac{1}{6G_E(0)}\frac{\partial G_E(k^2)}{\partial k^2}|_{k^2=0}, \tag{11}$$

where R_p is the proton charge radius [4]. For point - like proton the recoil correction to energy levels of order $(Z\alpha)^5$ is determined by both non - relativistic and relativistic electron momenta in the two - photon interaction operator $V_{2\gamma}$. The corresponding expression takes the form [1]:

$$\Delta E_{\rm nl} = \frac{\mu^3}{m_1m_2}\frac{(Z\alpha)^5}{\pi n^3}\left[\frac{2}{3}\delta_{l0}\ln\left(\frac{1}{Z\alpha}\right) - \frac{8}{3}\ln k_0(n,l) - \frac{1}{9}\delta_{l0} - \frac{7}{3}a_n - \frac{2}{m_2^2-m_1^2}\delta_{l0}\left(m_2^2\ln\frac{m_2}{\mu} - m_1^2\ln\frac{m_2}{\mu}\right)\right], \tag{12}$$

$$a_n = -2\left[\ln\frac{2}{n} + \left(1+\frac{1}{2}+...+\frac{1}{n}\right) + 1 - \frac{1}{2n}\right]\delta_{l0} + \frac{1-\delta_{l0}}{l(l+1)(2l+1)}. \tag{13}$$

Taking in mind the results (10), (12), we can express infrared finite contribution of two - photon diagrams to the Lamb shift of hydrogen - like atom in the kind:

$$\Delta E_{2\gamma}^{\rm Ls} = -\frac{\mu^3}{\pi n^3}\delta_{l0}(Z\alpha)^5\int_0^\infty \frac{dk}{k}V(k), \tag{14}$$

$$V(k) = \frac{2(F_1^2-1)}{m_1m_2} + \frac{8m_1[F_2(0)+4m_2^2F_1'(0)]}{m_2(m_1+m_2)k} + \frac{k^2}{2m_1^3m_2^3}\left[2(F_1^2-1)(m_1^2+m_2^2)+4F_1F_2m_1^2+3F_2^2m_1^2\right] + \tag{15}$$

$$+\frac{\sqrt{k^2+4m_1^2}}{2m_1^3m_2(m_1^2-m_2^2)k}\left\{k^2\left[2(F_1^2-1)m_2^2+4F_1F_2m_1^2+3F_2^2m_1^2\right]-8m_1^4F_1F_2+\frac{16m_1^4m_2^2(F_1^2-1)}{k^2}\right\}-$$

$$-\frac{\sqrt{k^2+4m_2^2}m_1}{2m_2^3(m_1^2-m_2^2)k}\left\{k^2\left[2(F_1^2-1)+4F_1F_2+3F_2^2\right]-8m_2^2F_1F_2+\frac{16m_2^4(F_1^2-1)}{k^2}\right\}.$$

To calculate the correction (14), we have used the dipole parameterization of proton form factors:

$$F_1(k^2) = \frac{G_E-\frac{k^2}{4m_2^2}G_M}{1-\frac{k^2}{4m_2^2}}, \quad F_2(k^2) = \frac{G_M-G_E}{1-\frac{k^2}{4m_2^2}}, \tag{16}$$

$$G_M = \frac{1+\kappa}{\left(1-\frac{k^2}{\Lambda^2}\right)^2}, \quad G_E = \frac{1}{\left(1-\frac{k^2}{\Lambda^2}\right)^2},$$

where proton structure parameter was taken as for calculation of hyperfine splitting in hydrogen atom $\Lambda = 0.898m_2$ [16], κ =1.792847 is the proton anomalous magnetic moment. Numerical value of correction (14) in the Lamb shift of electronic hydrogen $\Delta E_H^{\rm Ls}$ (2P-2S) = 4.25 Hz and of muonic hydrogen $\Delta E_{\mu p}^{\rm Ls}$ = 4.35 GHz, what coincides with the result of [5]. Let consider now the effects of proton polarizability, which are described by the dispersion integral (7). The imaginary parts of amplitudes $C_i(k_0,k^2)$ are related with deep inelastic structure functions $F_i(x,Q^2)$:

$$\frac{1}{\pi}ImC_1(x,Q^2) = \frac{F_1(x,Q^2)}{m_2}, \quad \frac{1}{\pi}ImC_2(x,Q^2) = \frac{F_2(x,Q^2)}{\nu}, \quad x = \frac{Q^2}{2m_2\nu}. \tag{17}$$

In the limit $Q^2 \to 0$ the following conditions must be fulfilled:

$$F_2 = O(Q^2), \quad \frac{F_1}{m_2} - \frac{F_2\nu}{Q^2} = O(Q^2). \tag{18}$$

They play important role for the parameterization of structure functions $F_i(x,Q^2)$ at low Q^2 [18]. Using relations (6) - (7) and transforming the integration over four - dimensional Euclidian space in the loop amplitudes

$$\int d^4k = 4\pi\int_0^\infty k^3dk\int_0^\pi Sin^2\phi d\phi, \quad k^0 = kCos\phi, \tag{19}$$

we can represent the proton polarizability contribution to the Lamb shift of muonic hydrogen in the kind:

$$\Delta E_{\rm pol}^{\rm Ls} = -\frac{16\mu^3(Z\alpha)^5m_1}{\pi^2n^3}\int_0^\infty\frac{dk}{k}\int_0^\pi Sin^2\phi d\phi\int_{\nu_0}^\infty dy\frac{1}{k^2+4m_1^2Cos^2\phi} \tag{20}$$

$$\frac{1}{(y^2+k^2Cos^2\phi)}\left[(1+2Cos^2\phi)\frac{\left(1+\frac{k^2}{y^2}\right)Cos^2\phi}{1+R(y,k^2)}+Sin^2\phi\right]F_2(y,k^2),$$

where $R(y,k^2) = \sigma_L/\sigma_T$ is the ratio of the cross sections for the transversally and longitudally polarized virtual photons respectively. To do numerical integration on the basis of (20) it is necessary to know the experimental data on structure function $F_2(x,Q^2)$ and the ratio $R(x,Q^2)$ in the wide region of variables

Q^2 and x [18]. We have used theoretical parameterization from [19], where structure function F_2 is decomposed into two terms F_2^P and F_2^R, corresponding to pomeron and reggeon exchanges:

$$F_2 = F_2^P + F_2^R, \quad F_2^r = \frac{Q^2}{Q^2+m_0^2} C_r(t) x_r^{a_r(t)} (1-x)^{b_r(t)}, \quad r = P, R, \quad \frac{1}{x_r} = \frac{2m_2\nu + m_r^2}{Q^2+m_r^2}. \tag{21}$$

Table. Proton polarizability contribution to the S-state energy levels of electronic hydrogen and muonic hydrogen.

	(e^-p^+) Hz	(μ^-p^+) μeV	[23]		[24]	
1S	-94	-144	-72	-100	-95	-136
2S	-11.8	-18	-9	-13	-11.9	-17

Recent experimental data on extraction of quantity $R(x,Q^2)$ from deep inelastic e-p scattering made at SLAC have lead to 6-parameter models for this ratio in a kinematic range $0.005 \le x \le 0.86$ and $0.5 \le Q^2 \le 130$ [20]:

$$R = \frac{a_1}{\ln(Q^2/0.04)} \Theta(x,Q^2) + \frac{a_2}{[Q^8+a_3^4]^{1/4}} \left[1 + a_4 x + a_5 x^2\right] x^{a_6}, \tag{22}$$

$$\Theta(x,Q^2) = 1 + 12 \left(\frac{Q^2}{Q^2+1}\right) \left(\frac{0.125^2}{0.125^2+x^2}\right).$$

In the case of muonic hydrogen the numerical value of Lamb shift (2P-2S) contribution, obtained on the basis of (20)-(22), is equal to

$$\Delta E_{\rm pol}^{\rm Ls} = 4.4 \pm 0.5 \ \ \text{GHz}, \tag{23}$$

For 1S - state of electronic hydrogen the correction on proton polarizability is -94 ± 10 Hz. This contribution is in good agreement with the results of [21, 22, 23, 24]. Displacements of 1S- and 2S-energy levels of electronic hydrogen and muonic hydrogen and the comparison with the results of [23, 24] are shown in table (the displacements of muonic hydrogen are expressed in eV for the convenience of comparison). It is necessary to point out that the used parameterization (21) satisfies the asymptotic condition (18) and leads to infrared finite contribution to the energy spectrum. So, the main uncertainty of our result is connected not with logarithmic approximation of the momentum integration similar to (20) as in [21, 22], but with experimental errors of $R(x,Q^2)$ in the range $Q^2 \le 0.5$ and experimental uncertainties of $F_2(x,Q^2)$ [18]. Correction (23), obtained by us for muonic hydrogen has the same order as other proton structure contributions (12) and (14). So, it has to be taken into account when extracting of proton charge radius R_p from future experimental measurement of muonic hydrogen Lamb shift.

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