Regular Representations of R-deformed Heisenberg Algebra

M. N. Mnatsakanova[1],
Skobeltsyn Institute of Nuclear Physics, Moscow State University,
Moscow, 119899, Russia

and

Yu. S. Vernov[2]
Institute for Nuclear Research, Russian Academy of Sciences,
Moscow, 117312, Russia

Abstract

All regular irreducible representations of R-deformed Heisenberg algebra are described in any weakly completed, non-degenerate inner product space.

R-deformed Heisenberg algebra (briefly R-algebra) is defined in the case of one degree of freedom by the following equations:

$$\left\{ \begin{array}{lllll} [a, a^+] & = & 1 + \nu R, & & \nu \in \mathbb{R} \\ \{a, R\} & = & \{a^+, R\} & = 0 & \\ R^2 & = & 1 & & \end{array} \right. \tag{1}$$

Operators a and a^+ are conjugated ones with respect to some inner product $< \cdot, \cdot >$. Such an algebra is connected with parafields and parastatistics, description of anions and so on. It was studied intensively in many papers (see [1] and references therein). Most of them are devoted to the representations of R-algebra in Hilbert space, but Plyushchay revealed very interesting finite representation in indefinite metric space [2].

It is well known that indefinite metric space is the basic space in covariant gauge theory [3-5]. Thus it is interesting to study systematically representations of R-algebra in indefinite metric space. Here we describe some class of such representations , which we call regular. We restrict ourself by consideration of irreducible representations only. To explain precise meaning of the word "regular" in our case we consider first the case of Heisenberg algebra, i.e. the case of CCR.

$$[a, a^+] = 1 \tag{2}$$

By definition representation of CCR in Hilbert space is regular, if it can be written in Weyl form and corresponding groups are continuous. Note, that Rellich-Dixmier conditions are equivalent to above mentioned ones. In accordance with von Neumann's uniqueness theorem any regular (irreducible) representation of CCR is unitary equivalent to Schrödinger one (see [6] for details). Let's recall, that Schrödinger representation of CCR is realized by operators: $a = \frac{1}{\sqrt{2}}\left(q + \frac{d}{dq}\right)$, $a^+ = \frac{1}{\sqrt{2}}\left(q - \frac{d}{dq}\right)$ in space $L_2(-\infty, \infty)$. An example of non-regular representation of CCR is the representation realized by the same operators in space $L_2(a, b)$. In fact, only regular representations are considered as relevant to quantum theory. If representation of CCR is regular, then operator $N = a^+ a$ has discrete spectrum and vice versa ($Sp\, N = \mathbb{N}$) Very useful condition of regularity, which is also valid in an indefinite metric space was obtained in [7]. It was formulated as the condition of operator $exp(i\, s\, N,)\ s \in \mathbb{R}$ existence (see [7] for mathematical details). The existence of this operator leads to the discreteness of $Sp\, N$. This result can be extended for more general algebras [8]. All these algebras we consider in some non-degenerate, weakly completed space. Let's remind, that an inner product space J is a non-degenerate one, if condition $x \perp J$ implies $x = 0$. The space J is weakly completed if the existence of $x \in J$ such that $< x_n, y > \to < x, y >$, $\forall y$ follows from the convergence of $< x_n, y >\ \forall y \in J$. Properties of indefinite metric space are described in books [9, 10].

Let's proceed to the precise definitions. First, we point out that nontrivial mathematical problems arise as operators a and a^+ are unbounded operators (see proof in [6], p.2). It implies that algebra under consideration can be realized not in all J space, but only on some dense domain of it (see mathematical

[1]E-mail address: mnatsak@theory.npi.msu.su
[2]E-mail address: vernov@theory.npi.msu.su

details in [7, 8]). Here first we consider some class of algebras and then show that R-algebra belongs to this class.

Definition 1. *Algebra A is called generalized Heisenberg algebra if it is determined by the following conditions:*

$$a^{+}a = \varphi(N) \qquad aa^{+} = \varphi(N+1), \tag{3}$$

where $\varphi(N)$ is some function. Operator N satisfies the conditions:

$$[N, a] = -a, \qquad [N, a^{+}] = a^{+} \tag{4}$$

Operators a and a^{+} are adjoint operators and operator N is selfadjoint with respect to some inner product $<\cdot,\cdot>$, which is indefinite in general case.

As operators a, a^{+} and N are unbounded we admit that relations (3) and (4) are fulfilled on some dense domain D. Evidently $D \subset D(N)$, $D \subset D(a)$, $D \subset D(a^{+})$. We also suppose the existence of dense domain $D' \subset D$, stable under the action of operators a and a^{+}. Below we consider only D' and so omit superscript $'$.

We consider conditions (4) as equivalent to the conditions

$$N\,a = a\,(N-1), \qquad N\,a^{+} = a^{+}\,(N+1) \tag{4'}$$

We also suppose that if for some vector ψ vector $a\,N\,\psi$ exists, then vector $N\,a\,\psi$ determined by eq. (4′) exists as well and vice versa.

The simplest case of this algebra is the Heisenberg algebra, $\varphi(N) = N$. Another well-known example of such an algebra is q-deformed Heisenberg algebra (see [11] and references therein). Below we shall see that R-algebra is a generalized Heisenberg algebra.

Definition 2. *Representation of generalized Heisenberg algebra is regular if there exists unitary operator $U(s)$ such that on D:*

$$U(s)\,D \subset D \tag{5}$$

$$U(s)\,U(t) = U(s+t) \tag{6}$$

$$\lim_{\Delta\to 0} \frac{U(\Delta) - I}{\Delta}\psi = i\,N\,\psi \tag{7}$$

It is sufficient to suppose that this limit exists in a weak sense [7]. To simplify the proof of Theorem (see below) we suppose that this limit is a strong limit under some Banach norm $\|\cdot\|$. In fact it is possible to show that any regular representation of algebra under consideration is equivalent to some regular representation in Krein space and so we can suppose the existence of Banach norm in the space under consideration without loss of generality (see the discussion about the notion of equivalence of representations and other subtle points in [7]). We also suppose that if $\lim_{\Delta\to 0} \frac{U(\Delta)-I}{\Delta}\psi$ exists, then $\psi \in D(N)$ and eq.(7) is valid.

¿From conditions (6) and (7) it follows that

$$\frac{d}{ds}U(s)\,\psi = i\,N\,U(s)\,\psi, \qquad \forall s \in \mathbb{R} \quad \text{and} \quad \psi \in D \tag{8}$$

In accordance with (6) and (4) we have on D:

$$\frac{d}{ds}\left(U(s)\,a\,U(-s)\right) = i\,U(s)\,[N, a]\,U(-s) = -i\,U(s)\,a\,U(-s) \tag{9}$$

Thus

$$U(s)\,a\,U(-s) = e^{is}a; \qquad U(s)\,a^{+}\,U(-s) = e^{-is}a^{+} \quad \text{on D} \tag{10}$$

According to eq.(10)

$$[U(2\pi n), a] = [U(2\pi n), a^{+}] = 0; \qquad n \in \mathbb{Z} \tag{11}$$

Now let's introduce the proper notion of irreducibility.

Definition 3. *A regular representation of A is said to be irreducible if any densely defined closed operator, which leaves D invariant and commutes with a and a^{+} is multiple of identity.*

It can be shown that this definition of irreducibility leads to the nonexistence of domain in J, which is not dense in J and is invariant under algebra A.

In accordance with this definition of irreducibility :

$$U\,(2\pi\, n) = e^{2\pi i\theta}; \qquad 0 \le \theta \le 1 \tag{12}$$

So

$$V\,(2\pi\, n) = I \quad \text{where} \quad V\,(s) = e^{-i\theta\, s}\, U\,(s) \tag{12'}$$

Now we can prove the Proposition 1.

Proposition 1. *If there exists operator $U\,(s)$ satisfying the conditions (5) - (7), then operator N has an eigenvector :*

$$N\,\psi_\lambda = \lambda\,\psi_\lambda \tag{13}$$

In order to prove this assertion first let's construct the eigenvector of $V\,(s)$. Let $\psi \in D$ be any vector. Then it is easy to see that $\phi_k = \int_0^{2\pi} V(s) e^{-iks}\psi\, ds$ is an eigenvector of $V\,(s)$ (if $\phi_k \ne 0$).

Indeed,

$$V\,(s)\,\phi_k = \int_0^{2\pi} V\,(s)\, e^{-iks'}\, V\,(s')\,\psi\, ds' =$$

$$\int_0^{2\pi} e^{-iks'}\, V\,(s+s')\,\psi\, ds' = e^{iks} \int_s^{2\pi+s} e^{-iks'}\, V\,(s')\,\psi\, ds' == e^{iks}\,\phi_k \tag{14}$$

The existence of this vector follows from the conditions (5) - (7) (proof see in [8]).

Eq.(14) implies the existence of $\lim\limits_{s\to 0}(V\,(s) - e^{isk})\,\phi_k$, so $\phi_k \in D\,(N)$ and

$$N\,\phi_k = (k+\theta)\,\phi_k \tag{15}$$

Owing to non-degeneracy of J, there exists such k_0 that $\phi_{k_0} \ne 0$ as in the opposite case for arbitrary vector ϕ

$$<\phi, \phi_k> = \int_0^{2\pi} <\phi, V\,(s)\psi>\, e^{-iks}\, ds = 0 \quad \forall k \tag{16}$$

As $V\,(2\pi) = 1$, function $T\,(s) \equiv <\phi, V\,(s)\,\psi>$ is a periodic function and eq.(16) leads to $T(s) \equiv 0$. Thus $<\phi, \psi> = 0 \quad \forall\phi$ and so $\psi = 0$ as the space under consideration is a non-degenerate one.

Proposition 2. *If Proposition 1 is fulfilled, then $Sp\, N$ consists of numbers $\lambda + k$, $k \in \mathbb{Z}$ and vectors $a^n\,\psi_\lambda$ and $(a^+)^n\,\psi_\lambda$ exhaust the set of eigenvectors of N.*

In fact, from the conditions (4′) it follows that

$$N\,\psi_{\lambda-1} = (\lambda-1)\,\psi_{\lambda-1}, \quad \psi_{\lambda-1} = a\,\psi_\lambda, \tag{17}$$

$$N\,\psi_{\lambda+1} = (\lambda+1)\,\psi_{\lambda+1}, \quad \psi_{\lambda+1} = a^+\,\psi_\lambda \tag{18}$$

Continuing this process we obtain:

$$N\,\psi_{\lambda-n} = (\lambda-n)\,\psi_{\lambda-n}, \quad \psi_{\lambda-n} = a^n\,\psi_\lambda \tag{19}$$

$$N\,\psi_{\lambda+n} = (\lambda+n)\,\psi_{\lambda+n}, \quad \psi_{\lambda+n} = (a^+)^n\,\psi_\lambda \tag{20}$$

Moreover, in accordance with (3)

$$a\,\psi_{\lambda+n} = aa^+\,\psi_{\lambda+n-1} = \varphi\,(\lambda+n)\,\psi_{\lambda+n-1},$$

$$a^+\,\psi_{\lambda-n} = a^+a\,\psi_{\lambda-n+1} = \varphi\,(\lambda-n+1)\,\psi_{\lambda-n+1}.$$

So

$$P_n\,(a, a^+) = \sum C_k\,\psi_{\lambda+k}, \qquad k \in \mathbb{Z}, \tag{21}$$

where $P_n(a, a^+)$ is an arbitrary polynomial of a and a^+. Thus $\psi_{\lambda+k}$ exhaust the set of eigenvectors of N in J_0, where J_0 is the space of all vectors $P_n(a, a^+)\,\psi_\lambda$. Our assertion is proved as we can consider J as closure of J_0.

Remark Let's point out, that for a given J_0 there exist different completions $\bar{J}_0$, corresponding to different Banach structure on J_0. But we can consider representations of the algebra in question in all these space as equivalent as they coincide on dense domain J_0 (see discussion of a similar problem in [7]). There exists a distinguished closure $\bar{J}_0$ (distinguished Krein structure on J_0), which can be constructed in the following way. First we construct the set of orthonormal vectors

$$e_k = \frac{\psi_{\lambda+k}}{| < \psi_{\lambda+k}, \psi_{\lambda+k} > |^{\frac{1}{2}}}$$

$$< e_k, e_k >= \delta_{\lambda k}\, \delta_k,$$

where $\delta_k = \pm 1$. Let's point out, that neutral vectors $\psi_{\lambda+k}$ are excluded owing to non-degeneracy of space J. Dividing the whole set of normed eigenvectors on the set of positive e_k^+ and negative e_k^- vectors, $< e_k^\pm, e_k^\pm >= \pm 1$ (see below eqs. (29), (31)), we see that

$$J_0 = J_0^+ \oplus J_0^-,$$

where $J_0^\pm$ are the spaces with positive and negative metrics. Complete $J_0^\pm$ using intrinsic scalar product $< \cdot, \cdot >$ and define

$$\bar{J}_0 = \bar{J}_0^+ \oplus \bar{J}_0^- \tag{22}$$

So we can say that representation in question is equivalent to the representation in Krein space $\bar{J}_0$.

Now let's proceed to R-algebra and show that R-algebra is a generalized Heisenberg algebra (see Definition 1). It is sufficient to show, that there exists operator N, satisfying the conditions (4) and commuting with R. Indeed, it can be directly checked, that operator

$$N_0 = 1/2\,\{a, a^+\} \tag{23}$$

satisfies all necessary conditions (see eqs. (3)). The general form of operator N is:

$$N = N_0 + C\,I \tag{24}$$

as we consider irreducible representation. In order that N be a^+a if $\nu = 0$ (CCR case) we choose $C = -\frac{\nu+1}{2}$.

As $[R, N] = 0$

$$R\,\psi_{\lambda+k} = R_{\lambda+k}\,\psi_{\lambda+k} \tag{25}$$

¿From the conditions $\{a, R\} = \{a^+, R\} = 0$, $R^2 = 1$ it follows that

$$R_{\lambda+k} = \pm\,(-1)^k \tag{26}$$

Then, directly

$$aa^+ = (N+1) + \frac{\nu}{2}\,(1+R) \tag{27}$$

$$a^+a = N + \frac{\nu}{2}\,(1-R) \tag{28}$$

Thus

$$< \psi_{\lambda+k+1}, \psi_{\lambda+k+1} >= \Big(\lambda + k + 1 + \frac{\nu}{2}\,(1 + R_{\lambda+k})\Big) < \psi_{\lambda+k}, \psi_{\lambda+k} > \tag{29}$$

$$< \psi_{\lambda+k-1}, \psi_{\lambda+k-1} >= \Big(\lambda + k + 1 + \frac{\nu}{2}\,(1 - R_{\lambda+k})\Big) < \psi_{\lambda+k}, \psi_{\lambda+k} > \tag{30}$$

Now it is easy to enumerate all kinds of $Sp\,N$:

1). $Sp\,N$ is bounded from below.

In this case there exists λ, such that

$$a\,\psi_\lambda = 0 \tag{31}$$

and $Sp\,N = \lambda + n$, $n \in \mathbb{N}$. It is easy to see, that eq.(31) implies that $\psi_{\lambda-1} = 0$, as we considered representation of R-algebra in a non-degenerate space.

In accordance with eqs. (29) and (31) we have in this case:

$$
\begin{aligned}
<\psi_{\lambda+k}, \psi_{\lambda+k}> = \Big(\lambda + k + \frac{\nu}{2}(1 + R_{\lambda+k-1})\Big)\,\Big(\lambda + k - 1 + \frac{\nu}{2}(1 + R_{\lambda+k-2})\Big) \cdots \\
\Big(\lambda + 1 + \frac{\nu}{2}(1 + R_{\lambda})\Big) <\psi_{\lambda}, \psi_{\lambda}>, \qquad k \in \mathbb{Z}^{+}
\end{aligned}
\tag{32}
$$

Owing to eq.(30) the condition (31) means that

$$
\lambda + \frac{\nu}{2}(1 - R_{\lambda}) = 0 \tag{33}
$$

Thus we have two possibilities:

i)

$$
\lambda = 0, \quad \text{and} \quad R\,\psi_{\lambda} = \psi_{\lambda} \tag{34}
$$

ii)

$$
\lambda = -\nu, \quad \text{and} \quad R\,\psi_{\lambda} = -\psi_{\lambda} \tag{35}
$$

2). $Sp\,N$ is bounded from above. It means that there exists $\tilde{\lambda}$ such that

$$
a^{+}\,\psi_{\tilde{\lambda}} = 0 \tag{36}
$$

Thus $Sp\,N = \tilde{\lambda} - n$, $n \in \mathbb{N}$.

$$
\begin{aligned}
<\psi_{\tilde{\lambda}+k}, \psi_{\tilde{\lambda}+k}> = \Big(\tilde{\lambda} + k + 1 + \frac{\nu}{2}(1 - R_{\tilde{\lambda}+k+1})\Big)\,\Big(\tilde{\lambda} + k + 2 + \frac{\nu}{2}(1 - R_{\tilde{\lambda}+k+2})\Big) \cdots \\
\Big(\tilde{\lambda} + \frac{\nu}{2}(1 - R_{\tilde{\lambda}})\Big) <\psi_{\tilde{\lambda}}, \psi_{\tilde{\lambda}}>, \qquad k \in \mathbb{Z}^{-}
\end{aligned}
\tag{37}
$$

Owing to eq.(29), condition (36) means that

$$
\tilde{\lambda} + 1 + \frac{\nu}{2}(1 + R_{\tilde{\lambda}}) = 0 \tag{38}
$$

We have again two possibilities:

i.

$$
\tilde{\lambda} = -1 \qquad R\,\psi_{\tilde{\lambda}} = -\psi_{\tilde{\lambda}} \tag{39}
$$

ii.

$$
\tilde{\lambda} = -(1 + \nu) \qquad R\,\psi_{\tilde{\lambda}} = \psi_{\tilde{\lambda}} \tag{40}
$$

3. finite spectrum.
There exist λ and $\tilde{\lambda}$ such that

$$
a\,\psi_{\lambda} = 0, \qquad a^{+}\,\psi_{\tilde{\lambda}} = 0 \tag{41}
$$

In accordance with eqs. (33) and (37) this is possible if

i.

$$
\lambda = 0; \quad \tilde{\lambda} = 2\,n \quad \nu = -(2\,n + 1) \tag{42}
$$

This representation has been revealed by Plyushchay.

ii.

$$
\lambda = -(1 + \nu); \quad \tilde{\lambda} = -1 \quad \nu = 2\,n + 1 \tag{43}
$$

This is a new representation.

4. unbounded spectrum. In this case

$$
a\,\psi_{\lambda+k} \neq 0, \quad a^{+}\,\psi_{\lambda+k} \neq 0 \quad \forall k; \quad Sp\,N = \lambda + k, k \in \mathbb{Z} \tag{44}
$$

Let's consider the correspondence between metric of space under question and kind of representations. In case when $Sp\,N$ is not bounded from below, metric is indefinite. This assertion follows directly from eq.(30). If $Sp\,N$ is finite, metric is also indefinite, excluding the trivial case of one dimensional representation (see eqs. (32) and (34)).

If $Sp\,N$ is bounded from below, then in case $\lambda = 0$ metric is positive if $\nu \geq -1$ and indefinite if $\nu < -1$. In case $\lambda = -\nu$ metric is positive if $\nu \leq 1$ and is indefinite if $\nu > 1$. These statements follow directly from eqs. (29), (32).

Conclusions Under general conditions of regularity we describe all irreducible representations of R-deformed Heisenberg algebra in any weakly completed, non-degenerate inner product space.

References

[1] M.S. Plyushchay, Nucl.Phys. B 491 (1997) 619.

[2] M.S. Plyushchay, Mod. Phys. Lett. A 11 (1996) 2953.

[3] F. Strocchi and A.S. Wightman, J. Math. Phys. 15 (1974) 2198.

[4] G. Morchio and F. Strocchi, Ann. Inst. Henri Poincaré, 33 (1980) 251.

[5] F. Strocchi, Selected Topics on the General Properties of Quantum Field Theory, World Scientific, 1993

[6] C.R. Putnam, Commutation Properties of Hilbert Space Operators and Related Topics, Springer, Berlin, 1967.

[7] M. Mnatsakanova, G. Morchio, F. Strocchi and Yu. Vernov, J. Math. Phys. 29 (1998) 2969.

[8] M.N. Mnatsakanova, G. Morchio and Yu.S. Vernov, *Proceedings of the International Seminar "Quarks-96"*, vol.2, page 51. Moscow, 1997.

[9] J. Bognar, Indefinite Inner Product Spaces, Springer-Verlag, Berlin, 1974.

[10] T.Ya. Azizov, I.S. Iokhvidov, Linear Operators in Spaces with an Indefinite Metric, J.Wiley and Sons, 1989 (in Russian: M.:Nauka, 1986).

[11] M. Chaichian, M.N. Mnatsakanova and Yu.S. Vernov, J.Phys. A: Math. Gen. 27 (1994) 2053.