On the Size of a Graviton

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Abstract. We propose an approximate wavefunction of the bound state of N $D0$-branes. Its spread grows as $N^{\frac{1}{3}}$ per particle, i.e. it saturates the Polchinski's bound.

1. Introduction

Definition of M-theory [1][2] using its ten dimensional compactification requires the knowledge of the behaviour of the Kaluza-Klein modes of the graviton. These are realized as the bound states [3] of $D0$-branes of Type IIA string theory, originally discovered as the black holes in the supergravity [4]. The aim of this paper is to get a good grip on the wavefunction Ψ_N of the boundstate of N $D0$-branes, for any N.

Of course, the quantum mechanics of N $D0$-branes contains sectors with N free particles, with several clumps of $D0$-branes bound to each other. In other words, for each partition $\vec{n}$:

$$N = 1 \cdot n_1 + 2 \cdot n_2 + \ldots + k \cdot n_k$$

there is a state whose wavefunction looks like:

$$\Psi_{\vec{n}} \sim \prod_{l=1}^{k} \Psi_l(x_{l,1})\Psi_l(x_{l,2}) \ldots \Psi_l(x_{l,n_l}) e^{ip_{l,1}\cdot x_{l,1} + \ldots ip_{l,n_l}\cdot x_{l,n_l}}$$

Our approximation contains all such states, which allows to compute the decay rates and overlaps between multiparticle states and single-particle states.

We are not going to review the arguments in favor of the existence of the bound state (see, e.g. [5][6][7][8][9][10][11]). We proceed with the variational approach to the problem of finding of the ground state. To do this we perturb the quantum mechanical problem: $\hat{H}_0 \to \hat{H}_{\mathbf{m}}$. The new problem with the Hamiltonian $\hat{H}_{\mathbf{m}}$ has only four supercharges instead of the sixteen which were the symmetries of $\hat{H}_0$. But the advantage is the better control on the spectrum of $\hat{H}_{\mathbf{m}}$.

We then take the approximate ground state of the perturbed model as the trial wavefunction for the original problem and minimize the bosonic contribution to the energy with respect to the parameter $\mathbf{m}$ of the perturbation. Of course the standard approach would be to minimize the full energy. But our trial wavefunctions are the ground states of the perturbed Hamitlonian so that we need to find another optimizing criterium. We suggest to look at the bosonic potential which is for large N parametrically equivalent to looking at the spread $\langle \mathrm{Tr} X^2 \rangle$ of the ground state. In this way we get for the optimal value of the parameter $|\mathbf{m}| \sim N^{-\frac{2}{3}}$.

As a general remark we would like to stress that our approach cannot be viewed as a completely satisfactory one, but nevertheless we consider it useful as it allows to study the large N case and serves as a nice complement to the asymptotic approach of [11] and subsequent work [12] (see [13] for the further apology for our point of view). Also it seems very simple to generalise our approach for the other gauge groups, rather then $SU(N)$ thus extending the work [14].

As one of the most interesting applications we suggest to compute the overlap between two-particle wavefunction with particles of mass N_1, N_2 and the single state with $N = N_1 + N_2$. We hope to return to this problem in the near future.

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2. Quantum mechanics

Consider the quantum mechanics of $D0$-branes in Type IIA theory in the flat Minkovski space. As it is well-known [3] in the sector with N particles they are described by the dimensional reduction of the $U(N)$ $\mathcal{N} = 4$ super-Yang-Mills theory down to $0 + 1$ dimensions (such gauge quantum mechanics with sixteen supersymmetries were introduced a long time ago [15], and for less number of supercharges in [16]). Upon excluding the center-of-mass motion it becomes $SU(N)/\mathbf{Z}_N$ gauge quantum mechanics.

Hamiltonian and Symmetries. The Hamiltonian of the model operates on the spinor wavefunctions. It is given by [15]:

$$\frac{1}{2}\mathrm{Tr} P_M P_M - \frac{1}{4}\mathrm{Tr}[X^M, X^N]^2 - \frac{1}{2}\mathrm{Tr}\psi\Gamma_M[X^M, \psi] \tag{2.1}$$

where all fields are traceless hermitian matrices, the indices M, N run from 1 to 9, repeated indices are summed over, in addition, ψ are the Majorana-Weyl fermions of $SO(9)$, Γ_M are the Dirac matrices of $SO(9)$.

The canonical commutation relations are:

$$[P_{M,A}, X^N_B] = \delta^N_M \delta_{AB}, \quad \{\psi_{\alpha,A}, \psi_{\beta,B}\} = \delta_{\alpha\beta}\delta_{AB} \tag{2.2}$$

where A, B are the $SU(N)$ indices.

The supersymmetry of this model is generated by the sixteen supercharges:

$$Q_\alpha = \Gamma^M_{\alpha\beta} \mathrm{Tr} P_M \psi_\beta - \frac{i}{4}[\Gamma^M, \Gamma^N]_{\alpha\beta} \mathrm{Tr}\psi_\beta [X^M, X^N] \tag{2.3}$$

It is sometimes convenient to think of this supersymmetric model as of the reduction of the four dimensional $\mathcal{N} = 4$ super Yang-Mills theory. In this formulation only the $S0(3)_E \times SO(6)_R$ part of the global symmetry group $SO(9)$ (or more presicely the cover of the that) is manifest. In our approach we break this symmetry. Namely we want to work in the $\mathcal{N} = 1$ four dimensional terms. The choice of $\mathcal{N} = 1$ subalgebra breaks $SO(6)_R$ down to $SU(3)_R$.

In the $\mathcal{N} = 1$ four dimensional terms the field content of the problem is that of a vector multiplet (A_μ, λ_α) and a triple of chiral multiplets $\left(q^i, \psi^i_\alpha\right)$, with the superpotential

$$W_0 = \frac{i}{6}\epsilon_{ijk} \mathrm{Tr} q^i [q^j, q^k] \tag{2.4}$$

The indices μ run from 0 to 3, $i = 1, 2, 3$ is the index in the $\mathbf{3}$ of $SU(3)_R$, $\alpha = 1, 2$ is the index in the $\mathbf{2}$ of the spin cover of $SO(3)_E$. The classical vacua are the minima of the associated potential

$$V = \sum_i \mathrm{Tr} F_i F_{\bar{i}} + \mathrm{Tr} D^2 + \sum_{m,i} \mathrm{Tr}[A_m, q^i][q^{i,\dagger}, A_m] - \sum_{m<n} \mathrm{Tr}[A_m, A_n]^2, \tag{2.5}$$

$m, n = 1, 2, 3$, where

$$\begin{aligned} F_i = \frac{\partial W_0}{\partial q^i}, \quad F_{\bar{i}} = F_i^\dagger, \\ D = \frac{1}{2} \quad \sum_i [q^i, q^{i,\dagger}] \end{aligned} \tag{2.6}$$

The vector multiplet fields transform in $(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1})$ of $SO(3)_E \times SU(3)_R$. The chiral multiplets fall in $(\mathbf{1}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{3})$.

Deformation. We are going to study the deformed quantum mechanical model, where the superpotential is replaced by:

$$W_{\mathbf{m}} = W_0 + \frac{1}{2}\mathbf{m} \mathrm{Tr} q^i q^i \tag{2.7}$$

This deformation breaks the $SO(9)$ global symmetry down to $SO(3)_E \times SO(3)_R$, where $SO(3)_R$ is imbedded into $SU(3)_R$ as a subgroup of unitary transformations preserving the 'metrics' δ_{ij}. The parameter of the deformation is the complex number $\mathbf{m}$.

The deformation (2.7) was succesfully used in four dimensions in the analysis of the topologically twisted $\mathcal{N}=4$ theory [17], in the problem of evaluating the index of the quantum mechanical problem [10], in the arguments in favor of the existence of the bound state [6] (for prime N).

The critical points of the deformed superpotential are the solutions to:

$$[q^j, q^k] = i\mathbf{m}\varepsilon_{ijk}q^i \tag{2.8}$$

i.e. $\frac{1}{\mathbf{m}}q^k$ must form a representation of the sl_2 Lie algebra. All finite-dimensional representations of sl_2 are unitary and decompose into sums of the irreducible representations:

$$\mathbf{N} = \mathbf{1}\cdot v_1 \oplus \mathbf{2}\cdot v_2 \ldots \oplus \mathbf{k}\cdot v_k, \tag{2.9}$$

v_l are the multiplicities. The minima of the potential (2.5) must in addition obey

$$[A_m, q^i] = 0, m = 1,2,3, \quad [A_m, A_n] = 0 \tag{2.10}$$

which implies that A_m must belong to the Lie algebra of

$$H = S\left(U(v_1)\times \ldots U(v_k)\right). \tag{2.11}$$

Although for generic choice of v_i's the only massless modes are those of gauge fields taking values in H, for the special choices of v_i one gets extra matter.

Let V be the original complex N-dimensional space (the space of Chan-Paton indices), $V_k = \mathbf{C}^{v_k}$ the multiplicity space, $\mathcal{R}_k \approx \mathbf{C}^k$ the standard spin $\frac{k-1}{2}$ representation of sl_2. Then (2.9) can be rewritten as equality of two reprsentations of sl_2:

$$V = \bigoplus_k V_k \otimes \mathcal{R}_k \tag{2.12}$$

Due to pseudo-reality of the representations of $SU(2)$ we have $V \approx V^\dagger$, although this isomorphism goes through the non-trivial transformation $L_i \to -L_i^\dagger$. Now all the fields of our quantum mechanical system can be written according to their sl_2 transformation properties.

The gauge field $A_m \in V\otimes V\otimes \mathcal{R}_1 \otimes \mathbf{R}^3$

The fermions $\psi_\alpha^i \in V\otimes V\otimes \mathcal{R}_3 \otimes \mathbf{C}^2$

The fermions $\lambda_\alpha \in V\otimes V\otimes \mathcal{R}_1 \otimes \mathbf{C}^2$

The scalars $q^i \in V\otimes V\otimes \mathcal{R}_1$

The importance of this representation is justified by the following statement: *In the quadratic approximation the Hamiltonian of the model is equal to:*

$$H_2 = \sum_{\text{fields}} \Phi^\dagger C_2 \Phi, \quad C_2 = \sum_i L_i L_i \tag{2.13}$$

The tensor product $V \otimes V$ always contains at least one component of spin zero. This is the trace part of the matrices which must be projected out for we deal with $SU(N)$ rather then $U(N)$ fields. Of course, if more then one v_k is different from zero then the spin zero components in the product $V \otimes V$ survive. In the problem of the computation of the index such choices of $\vec{v}$ contribute zero (see [10] for the principal contribution and [7] for the boundary terms).
It is perhaps worthwhile mentioning here that the set of massless modes coincide with the field content of the theory obtained by orbifolding of the $\mathcal{N} = 4$ super-Yang-Mills theory by the $SO(3)$ subgroup of $SU(3)$ in the spirit of [18][19][1]

The next section is devoted to the justification of the claim about the quadratic Casimir. Let L_i be the generators of $SU(2)$ in the representation V,

$$[L_i, L_j] = i\varepsilon_{ijk} L_k \tag{2.14}$$

Bosonic potential. Write:

$$\begin{aligned} q_i = \mathbf{m}\left(L_i + \xi_i\right), \quad q_i^\dagger = \bar{\mathbf{m}}\left(L_i + \xi_i^\dagger\right), \\ \xi_i = \beta_i + i\gamma_i, \quad A_m = |\mathbf{m}|\alpha_m \end{aligned} \tag{2.15}$$

with $\alpha_m, \beta_i, \gamma_i$ being Hermitian matrices. A little computation shows that the effective bosonic potential looks like:

$$\begin{aligned} \frac{1}{|\mathbf{m}|^4} V_\xi = \frac{1}{2}\sum_m \mathrm{Tr}\alpha_m C \alpha_m + \frac{1}{2}\sum_j \mathrm{Tr}\left(\beta_j(C+2)\beta_j + \gamma_j(C+2)\gamma_j\right) \\ - \sum_{i,j,k} \mathrm{Tr}\varepsilon_{ijk} L_k \left([\beta_j, \beta_i] + [\gamma_j, \gamma_i]\right) + \mathrm{Tr}\tilde{D}^2 \\ \tilde{D} = \sum_j [L_j, \beta_j] \\ CK = \sum_j [L_j, [L_j, K]] \end{aligned} \tag{2.16}$$

[1] The idea to orbifold by the infinite subgroups of $SO(6)$ arose in the discussion with C. Vafa and was dismissed by both of us as the crazy one. It still looks like the one but may prove to be useful.

Bosonic eigenmodes. The bosonic potential (2.16) gives masses to nine bosonic traceless hermitian matrices, provided that the gauge fixing is done appropriately. The point is that the critical points of the superpotential $W_{\mathbf{m}}$ form a continious family, due to the invariance of (2.8) under the complexified gauge transformations from $\mathrm{SL}_N(\mathbf{C})$. The non-compact part is broken by the presence of the square of the moment map term $\mathrm{Tr}\sum_i \left([q_i, q_i^\dagger]\right)^2$ but the compact part is not. The choice of the representatives L_i breaks the $SU(N)$ invariance but by the well-known Goldstone effect the gauge group reveals itself in the presence of the massless modes, of the form:

$$\beta_j = i[L_j, \chi], \quad \chi^\dagger = \chi \tag{2.17}$$

These massless modes are eventually killed by the Gauss law, so we might as well work in the gauge:

$$\tilde{D} = 0 \tag{2.18}$$

To see whether (2.18) is a good gauge choice let us make an infinitesimal gauge transformation *à la* (2.17). We see that $\tilde{D}$ changes by $iC\chi$. If the traceless part of $V \otimes V$ does not contain spin zero pieces then C is invertible and (2.18) is a good gauge choice. This is equivalent to the condition $v_k = \delta_{k,N}$, which is the case of our utmost interest.

Fermionic potential. The fermionic part of the Hamiltonian can be written in the following way (the indices α, β are raised and lowered with the help of $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ symbol, the indices i, j, k are raised with the help of δ_{ij}, we also have a $\delta_{i\bar{i}}$ pairing):

$$\begin{aligned}\frac{1}{2\sqrt{2}}\mathrm{Tr}\left(i\varepsilon_{ijk}q^i[\psi^j_\alpha, \psi^{k,\alpha}] + \mathbf{m}\psi_{i,\alpha}\psi^{i,\alpha} + q^{\bar{i}}[\psi^i_\alpha, \lambda^\alpha]\right) - \\ - \frac{1}{2\sqrt{2}}\mathrm{Tr}\left(i\varepsilon_{\bar{i}\bar{j}\bar{k}}q^{\bar{i}}[\bar{\psi}^{\bar{j}}_{\dot{\alpha}}, \psi^{\bar{k},\dot{\alpha}}] + \bar{\mathbf{m}}\bar{\psi}_{\bar{i},\dot{\alpha}}\bar{\psi}^{\bar{i},\dot{\alpha}} + q^i[\bar{\psi}^{\bar{i}}_{\dot{\alpha}}, \bar{\lambda}^{\dot{\alpha}}]\right)\end{aligned} \tag{2.19}$$

In the quadratic approximation it reduces to:

$$\frac{\mathbf{m}}{2\sqrt{2}}\mathrm{Tr}\left(iL_i[\psi^j_\alpha, \psi^{k,\alpha}]\varepsilon_{ijk} + \psi^i_\alpha\psi^{i,\alpha} - iL_i[\bar{\psi}_{\bar{i},\dot{\alpha}}, \bar{\lambda}^{\dot{\alpha}}]\right) + \text{c.c.} \tag{2.20}$$

Completely Higgsed phase. The case of our immediate interest is $v_k = \delta_{k,N}$, i.e. when the representation is irreducible. In this case the solution to (2.10) is $A_m = 0$ i.e. all fluctuations of A_m's are massive. In addition, since the irreps of $SU(2)$ have no moduli the fluctuations of q_i which are orthogonal to the gauge orbit are massive as well. Let L_i denote the standard generators of $SU(2)$ in the N-dimensional representaion:

$$\begin{aligned}L_3 =& \mathrm{diag}\,(-j, \ldots, j) \\ L_\pm =& L_1 \pm iL_2 \\ [L_\pm]_{m\pm1,m} =& \sqrt{(j+1\pm m)(j\mp m)}\end{aligned} \tag{2.21}$$

Then at the minimum of the potential $q_i = \mathbf{m}L_i, q_{\bar{i}}^\dagger = \bar{\mathbf{m}}L_i$, $A_m = 0$.

3. The size of the bound state

The zero energy eigen-function of the deformed Hamiltonian can be approximated by the Gaussian wavefunction, which is annihilated by the quadratic Hamiltonian (2.13) and is localised near the minimum of the potential given by the formulae (2.21). Let us estimate the bosonic spread of this wavefunction.

We want to evaluate

$$\Delta = \left\langle \sum_M \mathrm{Tr} X^M X^M \right\rangle = \tag{3.1}$$

$$\sum_m \langle \mathrm{Tr} A_m^2 \rangle + \sum_i \langle \mathrm{Tr} q_i q_{\bar{i}}^\dagger \rangle = |\mathbf{m}|^2 \sum_i \mathrm{Tr} L_i^2 + \frac{3}{2|\mathbf{m}|} \mathrm{Tr}_{V\otimes V - \mathcal{R}_0} \frac{1}{C^{\frac{1}{2}}} + \frac{1}{|\mathbf{m}|} \mathrm{Tr}_{V\otimes V\otimes \mathcal{R}_1 - \mathcal{R}_0} \frac{1}{C^{\frac{1}{2}}} =$$

$$= \frac{3}{4}|\mathbf{m}|^2 N(N^2-1) + \frac{9}{2|\mathbf{m}|} \sum_{l=1}^{N-2} \frac{2l+1}{\sqrt{l(l+1)}} + 4\frac{2N-1}{\sqrt{N(N-1)}} + 2\frac{2N+1}{\sqrt{N(N+1)}}$$

For N large Δ goes like:

$$\Delta \sim \frac{3}{4}|\mathbf{m}|^2 N^3 + 9\frac{N}{|\mathbf{m}|} \tag{3.2}$$

Let us now adjust $\mathbf{m}$ so as to minimize Δ. We expect to have small corrections both to the shape of the wavefunction and its spread for this value of the parameter $\mathbf{m}$ coming from the non-linear terms in the original Hamiltonian as well as from the added terms. We get:

$$\mathbf{m} \sim \frac{6^{\frac{1}{3}}}{N^{\frac{2}{3}}}, \qquad \Delta \sim 7.43 N^{\frac{5}{3}},$$

in agreement with the estimates [5][20][21].

Appendix A. Diagonalization of the effective mass matrix.

We want to diagonalize the effective mass operator:

$$\widehat{\mathcal{M}} : \phi \mapsto \sum_i [L_i, [L_i, \phi]] \tag{A.1}$$

In the basis where the generators of sl_2 are represented as (2.21) the eigen-value problem for the operator (A.1) $\widehat{\mathcal{M}}\phi = E\phi$ is written explicitly as:

$$\begin{aligned} &\sqrt{(j+1+m')(j-m')(j+1+m)(j-m)}\phi_{m+1,m'+1} + 2mm'\phi_{mm'} + \\ &+\sqrt{(j+m')(j+m)(j+1-m)(j+1-m')}\phi_{m-1,m'-1} = (2j(j+1) - E)\,\phi_{mm'} \end{aligned} \tag{A.2}$$

Reduction to the representation theory problem. Let V denote the N-dimensional representation of $su(2)$, so $\phi \in \mathrm{Hom}(V,V)$. In solving the eigen-value problem (A.2) we are allowed to perform the transformations $\phi \to g\phi h^{-1}$ where g, h are the group elements of $SU(2)$ acting in V. By applying such transformations we can map the operator $\widehat{\mathcal{M}}$ to the operator $\mathbf{M}$ acting in the space $V \otimes V$:

$$\mathbf{M} = \sum_{i=1}^{3} \left(L_i^2 \otimes 1 + 1 \otimes L_i^2 + 2L_i \otimes L_i \right) \tag{A.3}$$

Let $\hat{L}_i = L_i \otimes 1 + 1 \otimes L_i$ be the generator L_i acting in $V \otimes V$. Then

$$\mathbf{M} = \sum_i \hat{L}_i^2, \tag{A.4}$$

i.e. it is simply the quadratic Casimir acting in the tensor product. Clearly, $\mathbf{M}$ is diagonalized by decomposing $V \otimes V$ into irreducibles:

$$V \otimes V = \bigoplus_{l=0}^{N-1} \mathcal{R}_l, \quad \mathbf{M} = \oplus_{l=0}^{N-1} l(l+1)\mathbf{1}_{2l+1} \tag{A.5}$$

where $\mathcal{R}_l$ is the spin l irrep of $SU(2)$.

The zero mode corresponding to $l = 0$ is the center-of-mass degree of freedom, $\phi_0 \sim \mathbf{1}_N$ and is projected out by the condition that $\mathrm{Tr}\phi = 0$.

Polynomial representation I. Let us form the generating function:

$$\Phi(x,y) = \sum_{m,m'} \frac{\phi_{mm'} x^{j+m} y^{j+m'}}{\sqrt{(j+m)!(j-m)!(j-m')!(j+m')!}} \tag{A.6}$$

Then the operator $\widehat{\mathcal{M}}$ is represented by the first order bi-differential operator:

$$\widehat{\mathcal{M}} = 2j(2j+1) - (xy+1)^2 \partial_x \partial_y + 2j(1+xy)\left(x\partial_x + y\partial_y - 2j\right) \tag{A.7}$$

It is natural to pass to the coordinates $y, \rho = xy$. Then the solution to the eigen-value problem can be found in the separated form:

$$\Phi(x,y) = y^a \psi_a(\rho) \tag{A.8}$$

where for positive a the function $\psi_a(\rho)$ is a polynomial of degree $2j - a$, while for negative $a = -b < 0$ the function ψ_a has to be of the form $\psi_a(\rho) = \rho^b \tilde{\psi}_b(\rho)$ where $\tilde{\psi}_b(\rho)$ is a polynomial of degree $2j - b$. The function ψ_a is an eigen-function of the second-order differential operator:

$$2j - (\rho+1)^2 \left(\rho\partial_\rho^2 + (a+1)\partial_\rho\right) + 2j(\rho+1)(2\rho\partial_\rho + a) - 4j^2\rho \tag{A.9}$$

Let $t = 1 + \rho$. For $a \geq 0$ introduce generators of sl_2 (following the general method of [22]):

$$\begin{aligned} \ell_+ &= -t^2\partial_t + (2j-a)t \\ \ell_0 &= t\partial_t - j + \frac{a}{2} \\ \ell_- &= \partial_t \end{aligned} \tag{A.10}$$

Then the operator (A.9) can be written as:

$$\left(\ell_+ + \ell_0 - j - \frac{a}{2} - 1\right)\left(\ell_0 - j - \frac{a}{2}\right) \tag{A.11}$$

which is an upper-triangular matrix in the basis $1, t, \ldots, t^{2j-a}$ of the monomials, which form an invariant subspace for the operator (A.9) in the space of all polynomials (such operators are called quasi-exactly-solvable [22]) . Its diagonal entries are the eigen-values, hence the spectrum of $\widehat{\mathcal{M}}$ is given by:

$$E_{n,a} = (N-n)(N-n-1), \quad n = 0, \ldots, N-1-a, \quad a = 0, \ldots, N-1 \tag{A.12}$$

For $a < 0$ the spectrum is identical.

Polynomial representation Ⅱ. Let us form another generating function:

$$\tilde{\Phi}(x,y) = \sum_{m,m'} \frac{\phi_{mm'}(-x)^{j+m}y^{j-m'}}{\sqrt{(j+m)!(j-m)!(j-m')!(j+m')!}} \tag{A.13}$$

Then the operator $\widehat{\mathcal{M}}$ is represented by the first order bi-differential operator:

$$\widehat{\mathcal{M}} = 2j(2j+1) - (x-y)^2\partial_x\partial_y - 2j(x-y)\left(\partial_x - \partial_y\right) \tag{A.14}$$

It is natural to pass to the coordinates $t = \frac{x-y}{x+y}, \rho = x + y$. Then the solution to the eigen-value problem can be found in the separated form:

$$\tilde{\Phi}(x,y) = \rho^a\psi_a(t), \quad , 0 \leq a \leq 4j \tag{A.15}$$

where the function $\psi_a(t)$ is a polynomial of degree a. It is also an eigen-function of the second-order differential operator:

$$-\ell_+^2 + \left(\ell_0 + \frac{a}{2} - 2j\right)\left(\ell_0 + \frac{a}{2} - 2j - 1\right) \tag{A.16}$$

where we have introduced generators of sl_2:

$$\begin{aligned} \ell_+ &= -t^2\partial_t + at \\ \ell_0 &= t\partial_t - \frac{a}{2} \\ \ell_- &= \partial_t \end{aligned} \tag{A.17}$$

which make (A.16) an upper-triangular matrix in the basis $1, t, \ldots, t^a$ of the monomials. Its diagonal entries are the eigen-values, hence the spectrum of $\widehat{\mathcal{M}}$ is given by:

$$E_{n,a} = (N-n)(N-n-1), \quad n = 0, \ldots, a, \quad a = 0, \ldots, 2N-2 \tag{A.18}$$

For $a < 0$ the spectrum is identical.

Clebsch-Gordan coefficients. For reader's convenience we list here the relevant $3j$-symbols, quoting them from [23]:

$$\begin{pmatrix} j & j & l \\ m & -n & n-m \end{pmatrix} = \left(\frac{(j+m)!(j-m)!(j+n)!(j-n)!(l+m-n)!(l+n-m)!}{(2j-l)!(l!)^2(2j+l+1)!} \right)^{\frac{1}{2}} \times \sum_{k=0}^{2j-l} (-1)^{k+m+n} \begin{pmatrix} l \\ j-m-k \end{pmatrix} \begin{pmatrix} l \\ j-n-k \end{pmatrix} \begin{pmatrix} 2j-l \\ k \end{pmatrix} \tag{A.19}$$

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