# Massless Particles and the Geometry of Curves. Classical Picture

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**Abstract**

We analyze the possibility of description of $D-$dimensional massless particles by the Lagrangians linear on world-line curvatures $k_i$, $\mathcal{S} = \sum_{i=1}^{N} c_i \int k_i d\tilde{s}$. We show, that the nontrivial classical solutions of this model are given by space-like curves with zero $2N-$th curvature for $N \leq [(D-2)/2]$. Massless spinning particles correspond to the curves with constant ratio $k_{N+a}/k_{N-a}$. It is shown that only the system with action $\mathcal{S} = c \int k_N d\tilde{s}$ leads to irreducible representation of Poincaré group. This system has maximally possible number $(N+1)$ of gauge degrees of freedom. Its classical solutions obey the conditions $k_{N+a} = k_{N-a}$, $a = 1, \ldots, N-1$, while first $N$ curvatures $k_i$ remain arbitrary. This solution is specified by coinciding $N$ weights of the massless representation of little Lorentz group, while the remaining weights vanish.

## 1 Introduction

The search of Lagrangians, describing spinning particles, both massive and massless, has a long story. The conventional approach in this direction consists in the extension of the initial space-time $\mathbb{R}^{D-1.1}$ by the auxiliary odd/even coordinates equip a system with spinning degrees of freedom.

There is another, less developed approach, where the spinning particle systems are described by the Lagrangians, which are formulated in the initial space-time, but depend on higher derivatives. The aesthetically attractive point of the last approach is that spinning degrees of freedom are encoded in the geometry of its trajectories. The Poincaré and reparametrization invariance require the actions to be of the form

$$\mathcal{S} = \int \mathcal{L}(k_1, ...., k_N) d\tilde{s}, \tag{1.1}$$

where $k_I$ denote the reparametrization invariants (extrinsic curvatures) of curves $(0 < I \leq D-1)$, $d\tilde{s}$ denotes (pseudo)arch length:

$$d\tilde{s} = \begin{cases} |d\mathbf{x}| & \text{for non} - \text{isotropic curves} \\ |d^2\mathbf{x}|^{1/2} & \text{for isotropic curves} \end{cases} . \tag{1.2}$$

Various systems of this sort, depending on the first and the second curvatures of path in $\mathbb{R}^{3.1}$ and $\mathbb{R}^{2.1}$ are known (see [1, 2, 3, 5] and refs therein). Nevertheless, the only system, which leads to an irreducible representation of Poincaré group, is the model in $\mathbb{R}^{3.1}$, given by the action[1]

$$\mathcal{S} = c \int k_1 d\tilde{s},$$

which describes a massless spinning particle with helicity $c$ (which, upon quantization, may take arbitrary integer or half integer values). Surprisingly, this model has $W_3$ gauge symmetry [3], and is specified by the classical trajectories which are *space-like plane curves* with arbitrary first curvature: $k_1 = \forall$, $k_2 = 0$.

All other three- and four dimensional models lead to the reducible representations of Poincaré group. For example, the analog of Plyushchay's model on isotropic curves describes massive spinning particles with Majorana-like spectrum [5],

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$$\text{Spin} \times \text{Mass} = c^2/4.$$

The subject of this work is to analyze the problem:

*Can massless spinning particles in $D > 4$ dimensional space-time be described by the actions (1.1)?* In other words, do exist the actions (1.1), which generate the constraints, corresponding to the massless irreducible representations of Poincaré group [6, 7]?

For this purpose we perform the classical investigation of a $D$-dimensional massless (due to explicit scale invariance) model on non-isotropic curves given by the action

$$\mathcal{S} = \sum_{i=1}^{N} c_i \int k_i d\tilde{s}, \quad c_N \neq 0, \quad [c_i] = [\hbar], \tag{1.3}$$

where

$$d\tilde{s} = |d\mathbf{x}|, \quad k_I = \frac{\sqrt{\det \hat{g}_{I+1} \det \hat{g}_{I-1}}}{\det \hat{g}_I}, \qquad (g_I)_{ij} \equiv \frac{d^i\mathbf{x}}{(d\tilde{s})^i}\frac{d^j\mathbf{x}}{(d\tilde{s})^j}, \quad i,j = 1,\ldots,I. \tag{1.4}$$

We restrict ourselves to the Lagrangians linear on curvatures, since they are specified by the maximally possible (for given $N$) set of primary constraints [8]. Therefore, they are the only candidates to describe the massless particle systems, corresponding to irreps.

We establish the following interesting properties of the model (1.3):

- only the systems with
$$N \leq N_0 = [(D-2)/2],$$
admit nontrivial classical solutions ($N_0$ is the rank of the little Lorentz group);
- the classical solutions of the model under consideration are space-like curves with $k_{2N}$=0; the solutions, corresponding to irreps, are specified by constant $k_{N+a}/k_{N-a}$ ratio, $a = 1,\ldots,N-1$;
- the solutions, corresponding to reducible representations, always exist, if $\sum_{i=1}^{N} |c_i| > |c_N|$.

On the other hand, there is special case of (1.3), exceptional from many points of view, given by the action

$$\mathcal{S} = c \int k_N d\tilde{s}, \quad N \leq N_0 = [(D-2)/2] \tag{1.5}$$

The remarable properties of the model (1.5) are the following:

- the solution of the system (1.5) leads to the irreducible representation of Poincaré group. This solution is specified by the following weights of the little Lorentz group: $m_1 = m_2 = \ldots = m_N = c$, $m_{N+1} = \ldots = m_{N_0} = 0$.
- the model (1.5) has $N+1$ gauge degrees of freedom, corresponding, probably, to the classical limit of the $W_{N+2}$ algebra.
- the classical solution of this model is a space-like curve specified by the relations: $k_1, \ldots k_N$ are arbitrary; $k_{N+1} = k_{N-1}, \ldots, k_{N+N-1} = k_1$, $k_{2N} = 0$.

The paper is arranged as follows.
In *Section 2* we give the Hamiltonian formulation of the system (1.3) and analyze its general properties. In *Section 3* we present the complete set of constraints for the model (1.5) and for the models given by (1.3), where $N = 2, 3$ $c_1 c_N \neq 0$ and $N \geq 3$, $c_{i-2} = 0$, $c_{N-1}c_N \neq 0$, $\quad i = 3,\ldots N,$.

# 2 Hamiltonian Formulation

In order to obtain the Hamiltonian formulation of the system with the action (1.3) we have to replace it by the classically equivalent one, which depends on the first-order derivatives, and then perform the Legendre transformation. For this purpose it is more convenient to use (instead of explicit expressions

(1.4)) the recurrent equations for curvatures, which follow from the Frenet equations for moving frame $\{\mathbf{e}_a\}$:

$$\dot{\mathbf{x}} = s\mathbf{e}_1, \quad \dot{\mathbf{e}}_a = sK_a{}^b\mathbf{e}_b, \quad \mathbf{e}_a\mathbf{e}_b = \eta_{ab}, \tag{2.1}$$

$$K_{\dot{a}b} + K_{\dot{b}a} = 0, \quad K_{\dot{a}b} = \left\{ \begin{array}{ccc} \pm k_a, & \text{if} & b = a \pm 1 \\ 0, & \text{if} & b \neq a \pm 1 \end{array} \right. , \quad k_a \geq 0. \tag{2.2}$$

In the Euclidean space the Frenet equations read

$$\dot{\mathbf{x}} = s\mathbf{e}_1, \quad \dot{\mathbf{e}}_a = sk_a\mathbf{e}_{a+1} - sk_{a-1}\mathbf{e}_{a-1}, \qquad \mathbf{e}_0 = \mathbf{e}_{D+1} \equiv 0, \;\; k_0 = k_D = 0. \tag{2.3}$$

Consequently, we get

$$s = |\dot{\mathbf{x}}| = \dot{\mathbf{x}}\mathbf{e}_1, \quad sk_i = \dot{\mathbf{e}}_i\mathbf{e}_{i+1} = \sqrt{\dot{\mathbf{e}}_{\mathbf{i}}^2 - (sk_{i-1})^2}. \tag{2.4}$$

It is easy to verify, that for the transition to the Frenet equations for non-isotropic curves in the pseudo-Euclidean space, we have to substitute,

$$(\mathbf{e}_{\underline{a}},\; sk_{\underline{a}},\; sk_{\underline{a}-1}, s) \to (i\mathbf{e}_{\underline{a}},\; isk_{\underline{a}},\; isk_{\underline{a}-1}, (-i)^{\delta_{1\underline{a}}}s) \tag{2.5}$$

for some index $\underline{a}$.
The choice $\underline{a} = 1$ means the transition to time-like curve, while $\underline{a} = 2, \ldots, D$- to space-like ones. By this reason, through out the paper we use the Euclidean signature.

Taking into account the expressions (2.3),(2.4) one can replace the initial Lagrangian (1.3) (in arbitrary time parametrization $d\tilde{s} = sd\tau$, $s = |\dot{\mathbf{x}}|$) by the following one

$$\begin{array}{rl} \mathcal{L} & = s\sum_i c_{i-1}k_{i-1} + c_N\sqrt{\dot{\mathbf{e}}_N^2 - (sk)_{N-1}^2} + \mathbf{p}(\dot{\mathbf{x}} - s\mathbf{e}_1) + \sum_i \mathbf{p}_{i-1}(\dot{\mathbf{e}}_{i-1} - sk_{i-1}\mathbf{e}_i + sk_{i-2}\mathbf{e}_{i-2}) \\ & \qquad -s\sum_{i,j} d_{ij}\,(\mathbf{e}_i\mathbf{e}_j - \delta_{ij}) \end{array} \tag{2.6}$$

where $s, k_{i-1}, d_{ij}, \mathbf{p}_{i-1}, \mathbf{e}_i$ are independent variables, $k_0 = 0, \mathbf{p}_0 = \mathbf{e}_0 = 0$.

Performing the Legendre transformation for this Lagrangian (see for details [8]), one get the Hamiltonian system with the Hamiltonian structure

$$\begin{array}{c} \omega_N = d\mathbf{p} \wedge d\mathbf{x} + \sum_{i=1}^N d\mathbf{p}_i \wedge d\mathbf{e}_i, \\ \mathcal{H} = s\left[\mathbf{p}\mathbf{e}_1 + \sum_{i=1}^N k_{i-1}(\phi_{i-1.i} - c_{i-1}) + \frac{k_N}{2c_N}(\Phi_{N.N} - c_N^2) + \sum_{i,j=1}^N d_{ij}(\mathbf{e}_i\mathbf{e}_j - \delta_{ij})\right], \end{array} \tag{2.7}$$

and the primary constraints

$$\mathbf{p}\mathbf{e}_1 \approx 0, \tag{2.8}$$

$$\mathbf{e}_i\mathbf{e}_j - \delta_{ij} \approx 0, \tag{2.9}$$

$$\mathbf{p}_N\mathbf{e}_N \approx 0, \;\; \mathbf{p}_N\mathbf{e}_{i-2} \approx 0, \tag{2.10}$$

$$\phi_{i-1.i} \equiv \mathbf{p}_{i-1}\mathbf{e}_i - \mathbf{p}_i\mathbf{e}_{i-1} \approx c_{i-1}, \quad \Phi_{N.N} \equiv \mathbf{p}_N^2 - \sum_{i=1}^N (\mathbf{p}_N\mathbf{e}_i)^2 \approx c_N^2. \tag{2.11}$$

Notice that in this formulation $s$ and $sk_i$ play the role of Lagrangian multipliers, so that stabilization of primary constraints generates either secondary ones, or the explicit relations on the first $N$ curvatures.

It is convenient to introduce the new variables, instead of $\mathbf{p}_i$,

$$\begin{array}{c} \mathbf{p}^{\perp}{}_i \equiv \mathbf{p}_i - \sum_{j=1}^N (\mathbf{p}_i\mathbf{e}_j)\mathbf{e}_j, \quad \mathbf{p}_i^{\perp}\mathbf{p}_j^{\perp} \equiv \Phi_{i.j}, \quad \mathbf{p}^{\perp}{}_i\mathbf{e}_j = 0, \\ \phi_{i.j} \equiv \mathbf{p}_i\mathbf{e}_j - \mathbf{p}_j\mathbf{e}_i, \\ \chi_{ij} = \mathbf{p}_i\mathbf{e}_j, \quad i \geq j. \end{array} \tag{2.12}$$

Since the constraints $u_{ij}$ are conjugated to $\chi_{ij}$ and commute with $\mathbf{p}_i^{\perp}$ and $\phi_{ij}$, we can impose, without loss of generality, the gauge conditions $\chi_{ij} \approx 0$ fixing the values of $d_{ij}$

$$\chi_{ij} \approx 0 \;:\; \Rightarrow \quad 2d_{i.j} = \delta_{ij}(k_ic_i - k_{i-1}c_{i-1}). \tag{2.13}$$

In these variables the equations of motion (in proper-time gauge $s = 1$) read

$$\left\{\begin{array}{c} \dot{\mathbf{x}} = \mathbf{e}_1, \\ \dot{\mathbf{e}}_{i-1} = k_{i-1}\mathbf{e}_i - k_{i-2}\mathbf{e}_{i-2}, \\ \dot{\mathbf{e}}_N = k_N \mathbf{p}^{\perp}{}_N/c_N - k_{N-1}\mathbf{e}_{N-1}, \\ \dot{\mathbf{p}^{\perp}}{}_i = -\delta_{1.i}\mathbf{p} - k_{i-1}\mathbf{p}^{\perp}{}_{i-1} + k_{i+1}\mathbf{p}^{\perp}{}_{i+1} - k_N(\Phi_{i.N}\mathbf{e}_N - \phi_{i.N}\mathbf{p}^{\perp}_N)/c_N \\ \dot{\mathbf{p}} = 0 \\ \dot{\phi}_{i.j} = -k_{i-1}\phi_{i-1.j} + k_i\phi_{i+1.j} + -k_{j-1}\phi_{i.j-1} + k_j\phi_{i.j+1} - \frac{k_N}{c_N}\delta_{N[i}\Phi_{j].N} \end{array}\right. \tag{2.14}$$

The Poincaré generators of the system take the form

$$\mathbf{P} = \mathbf{p}, \quad \mathbf{M} = \mathbf{p}\times\mathbf{x} + \sum_{i=1}^{N}\mathbf{p}_i\times\mathbf{e}_i = \mathbf{p}\times\mathbf{x} + \sum_{i=1}^{N}\mathbf{p}^{\perp}{}_i\times\mathbf{e}_i - \frac{1}{2}\sum_{i,j=1}^{N}\phi_{ij}\mathbf{e}_i\times\mathbf{e}_j. \tag{2.15}$$

Now let us construct the secondary constraints.
Stabilization of the constraint (2.8) generates the following set of constraints

$$\mathbf{p}\mathbf{e}_i \approx 0, \quad \mathbf{p}\mathbf{p}_i \approx 0, \quad \mathbf{p}^2 \approx 0, \tag{2.16}$$

which provide for the mass-shell and transversality conditions. All the secondary constraints produced by the primary ones (2.11) are the functions of $\phi_{ij}$ and $\Phi_{ij}$, because these functions commute with $u_{ij}$,(2.8) and form closed algebra. One can arrange these functions in the matrices

$$\hat{R}^p_{ij} = (\phi^p_{ij}, \Phi^p_{ij}), \quad \phi^p_{ij} \equiv \phi_{i.i+p}, \ \ \Phi^p_{i.j} \equiv \Phi_{i.2N+1-i-p}, \quad p = 1,\ldots,N,$$

so that the secondary constraints of $(p+1)-$th stage, depend on $\hat{R}^{p\pm1}$. Thus stabilization procedure contains at most $N$ stages.

However, the choice of secondary constraints is not uniquely defined for any Lagrangian if $N > 2$. The primary constraints (2.11) generate the following first-stage secondary constraints,

$$\Phi_{N-1.N} \approx 0, \quad \phi_{i-2.i} \approx 0, \quad i = 3,\ldots,N. \tag{2.17}$$

At the next stage we get the system of linear equations on $k_i$:

$$\left\{\begin{array}{c} k_{i-4}\phi_{i-4.i-1} - k_{i-3}c_{i-2} + k_{i-2}c_{i-3} - k_{i-1}\phi_{i-3.i} \approx 0, \quad i = 3,\ldots,N \\ -k_{N-3}c_N\phi_{N-3.N} + k_{N-2}c_Nc_{N-1} - k_{N-1}c_Nc_{N-2} + k_N\Phi_{N.N-2} \approx 0 \\ k_{N-2}\Phi_{N-2.N} + k_{N-1}(\Phi_{N-1.N-1} - c_N^2) + k_Nc_Nc_{N-1} \approx 0. \end{array}\right. \tag{2.18}$$

So, only the system with the action (1.5) give rize to uniquely defined second-stage secondary constraints.

*What is the rule for choosing the secondary constraints which we have to follow in order to obtain the solution corresponding to massless spinning particle?*
¿From (2.16) it is seen that $N \le [(N-2)/2]$, while the helicity matrix is of the form

$$\mathbf{S} = \sum_{i=1}^{N}\mathbf{p}^{\perp}{}_i\times\mathbf{e}_i - \frac{1}{2}\sum_{i,j=1}^{N}\phi_{ij}\mathbf{e}_i\times\mathbf{e}_j, \quad \mathrm{tr}\ \mathbf{S}^{2i} = \mathrm{tr}\begin{pmatrix} 0 & 1 \\ -\hat{\Phi} & \hat{\phi}\end{pmatrix}^{2i}. \tag{2.19}$$

Therefore, only the solutions containing $N$ stabilization stages correspond to irreps. To get a massless spinning particle system, we have to choose the constraints which lower the rank of the system of the equations linear in $k_i$, and are compatible with the conditions $k_i \neq 0$.

¿From the equations of motion (2.14) one can see that space-like vectors $(\mathbf{e}, \mathbf{p}^{\perp}_i)$ define first $2N$ (non-normalized) elements of moving frame, while $\mathbf{p}$ defines its $(2N+1)-$th, isotropic element. One can orthonormalize the vectors $\mathbf{p}^{\perp}{}_i$, introducing [9]

$$\mathbf{e}_{N+1} \equiv \mathbf{p}^{\perp}_N/c_N, \quad \mathbf{e}_{2N+1-a} = -\frac{1}{\sqrt{\Delta_a\Delta_{a+1}}}\begin{vmatrix} \Phi_{N.N} & \cdots & \Phi_{N.a} \\ \vdots & & \vdots \\ \Phi_{a+1.N} & \cdots & \Phi_{a+1.a} \\ \mathbf{p}^{\perp}{}_N & \cdots & \mathbf{p}^{\perp}{}_a, \end{vmatrix}, \tag{2.20}$$

where $\Delta_N = 1$, $\Delta_a = \det \Phi_{\alpha.\beta}$, $\alpha, \beta = a, \dots N$, $a = 1, \dots, N-1$.
Comparing the equations of motion with (2.3), we get the following relations on curvatures

$$k_{2N} = 0, \quad k_{N+a} = \sqrt{\frac{\Delta_{N-a}\Delta_{N-a+2}}{\Delta^2_{N-a+1}}} k_{N-a}. \tag{2.21}$$

Summarizing the results obtained in this Section we conclude

**Proposition.** The systems with the actions (1.3) admit nontrivial classical solutions if $N \leq [(D-2)/2]$. These solutions are space-like curves with zero $2N-$th extrinsic curvature. The curves with constant ratio $k_{N+a}/k_{N-a}$ correspond to the massless particles with fixed helicities (the classical analogs of irreps). The only system, whose solutions corresponds to irreps, is defined by the action (1.5).

# 3 Examples

In this Section we consider the explicit examples of the systems defined by the action (1.3).

**Example 1:** $\mathcal{L} = ck_N$.
We start with the basic example, given by the action (1.5), whose solutions correspond to the massless irreducible representations of Poincaré group.

Primary constraints (2.11) generate the maximally possible set ($N^2$) of constraints, all of them are of the first-class,

$$\phi_{ij} \approx 0, \quad \Phi_{ij} - c^2\delta_{ij} \approx 0. \tag{3.1}$$

The lagrangian multipliers $k_i$ remain arbitrary, hence the system has $N+1$ gauge degrees of freedom. This is in correspondence with the conjecture of [4] that the gauge symmetries of the action (1.5) define the classical limit of $W_{N+2}$ algebra. Taking into account (2.21), we conclude that classical solution of the system under consideration is given by space-like curve specified by the conditions

$$k_1, \dots, k_N \text{ are arbitrary}, \quad k_{N+a} = k_{N-a}, \quad a = 1, \dots, N-1, \quad k_{2N} = 0.$$

The dimension of phase space is

$$\mathcal{D}^{phys} = 2(D-1) + N(2D - 3N - 5),$$

so,

$$N_0(2D - 3N_0 - 5) = \begin{cases} (D-2)(D-4)/4, & \text{for even } D, \\ (D-1)(D-5)/4, & \text{for odd } D. \end{cases} \tag{3.2}$$

Let us introduce the complex variables

$$\mathbf{z}_i = (\mathbf{p}_i + \imath c\mathbf{e}_i)/\sqrt{2}$$

in which the Hamiltonian system reads

$$\begin{gathered} \omega = d\mathbf{p} \wedge d\mathbf{x} + \tfrac{\imath}{c}\textstyle\sum_i d\mathbf{z}_i \wedge d\bar{\mathbf{z}}_i, \\ \mathcal{H} = \tfrac{s}{2c}\left[\imath\sqrt{2}\mathbf{p}(\bar{\mathbf{z}}_1 - \mathbf{z}_1) + \imath\textstyle\sum_{a=1}^{N-1} k_a(\mathbf{z}_a\bar{\mathbf{z}}_{a+1} - \mathbf{z}_{a+1}\bar{\mathbf{z}}_a) + k_N(\mathbf{z}_N\bar{\mathbf{z}}_N - c^2)\right], \end{gathered} \tag{3.3}$$

while the constraints take the conventional form

$$\begin{gathered} \mathbf{z}_i\bar{\mathbf{z}}_j - c^2\delta_{ij} \approx 0, \\ \mathbf{z}_i\mathbf{z}_j \approx 0, \\ \mathbf{p}\mathbf{z}_i \approx 0, \\ \mathbf{p}^2 \approx 0. \end{gathered} \tag{3.4}$$

The eigenvalues of the helicity matrix $\mathbf{S}$, $rank\mathbf{S} = N$, are given by the relations tr $\mathbf{S}^{2i} = c^{2i}$, $i = 1, \dots N$. So, the system is specified by the following weights $m_I$ of the little Lorentz group:

$$m_1 = \dots = m_N = c, \quad m_{N+1} = \dots = m_{N_0} = 0.$$

For $N = N_0$ this solution possesses conformal symmetry[2][10].

**Example 2:** $\mathcal{L} = c_1 k_1 + c_2 k_2, c_1 \neq 0.$
For this system the constraints (2.11) produce the secondary constraint $\Phi_{1.2} \approx 0$ and the relation on the curvatures

$$k_2 c_2 c_1 + k_1(\Phi_{1.1} - c_2^2) = 0,$$

The dimension of the phase space is $\mathcal{D} = 2(3D-10)$, while the value of $\Phi_{1.1}$ remains unfixed (this function defines the constant of motion).

Let us denote $\Phi_{1.1} \equiv c^2$. Then the curvatures obey the conditions

$$k_2 c_2 c_1 = k_1(c_2^2 - c^2), \quad c k_3 = c_2 k_1, \quad k_4 = 0$$

while the helicities are defined by the expressions

$$\mathrm{tr}\mathbf{S}^{2i} = (c_2^2 - c_1^2)^i + (c^2 - c_1^2)^i, \quad i = 1, 2.$$

**Example 3:** $\mathcal{L} = \sum_{i=1}^{3} c_i k_i$, $c_1 c_3 \neq 0$.
The primary constraints (2.11) produce the following secondary constraints of the first and second stage

$$\Phi_{2.3} \approx 0, \quad \phi_{1.3} \approx 0, \quad \Phi_{1.2} \approx 0,$$

and the relations on the lagrangian multipliers

$$\hat{A}\hat{k} \equiv \begin{pmatrix} c_3 c_2 & -c_3 c_1 & \Phi_{1.3} \\ \Phi_{1.3} & \Phi_{2.2} - c_3^2 & c_3 c_2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = 0 \tag{3.5}$$

When $\mathrm{rank}\hat{A} = 2$ the system has no other secondary constraints, and the helicities of the system are unfixed.

However, if $c_1 c_2 c_3 \neq 0$, one can lower the rank of $\hat{A}$ choosing

$$\Phi_{2.2} = c_3^2 \mp c_1 c_3, \quad \Phi_{1.3} = \pm c_2 c_3, \tag{3.6}$$

so the only preserved relation on curvatures is

$$c_2(k_1 \pm k_3) - c_1 k_2 = 0. \tag{3.7}$$

Stabilizing the constraints (3.6) we get

$$\Phi_{1.2} \approx 0, \quad (\Phi_{1.1} - c_3^2 \pm c_1 c_3) k_1 \pm c_2 c_3 k_2 = 0 \tag{3.8}$$

Thus, the function $\Phi_{1.1}$, being the constant of motion, remains arbitrary, and we have *two* relations on curvatures.

**Example 4:** $\mathcal{L} = c_1 k_{N-1} + c_2 k_N$, $c_1 \neq 0$, $N > 2$.
For this system we consider only those sets of constraints which define the solutions, corresponding to the irreps.

The constraints (2.11) produce the first-stage secondary constraints (2.17). Then we get the following set of the second-stage secondary constraints

$$\phi_{i-3.i} \approx 0,$$

and the relations on the lagrangian multipliers

$$\hat{A}\hat{k} = \begin{pmatrix} c_2 c_1 & 0 & \Phi_{N-2.N} \\ \Phi_{N-2.N} & \Phi_{N-1.N-1} - c_2^2 & c_2 c_1 \end{pmatrix} \begin{pmatrix} k_{N-2} \\ k_{N-1} \\ k_N \end{pmatrix} = 0.$$

[2]The author thanks M.Vasiliev for this remark

To get the solutions corresponding to irreps, we have to choose the constraints, which lower the rank of the matrix $\hat{A}$, namely

$$\Phi_{N-1.N-1} \approx c_2^2, \quad \Phi_{N.N-2} \approx -c_2 c_1.$$

Hence the only preserved relation on $k_i$ is $k_{N-2} = k_N$.

Continuing stabilization procedure, we get, finally

$$\begin{cases} \phi_{\alpha.\beta} \approx c_1 \epsilon_{N-\alpha.N-\beta}, \quad \phi_{\alpha.b} \approx 0 \;\; \phi_{a.b} \approx 0, \\ \Phi_{\alpha.\beta} \approx c_2^2 \delta_{\alpha\beta}, \;\; \Phi_{\alpha.b} \approx -c_2 c_1 \delta_{\alpha.b+2}, \\ \Phi_{a.b} \approx (c_2^2 - c_1^2)\delta_{ab} - c_2 c_1 \delta_{a.b\pm 2}, \\ k_i = k_{i-2}, \end{cases} \tag{3.9}$$

where $a, b = 1, \ldots N-2;$, $\alpha, \beta = N-1, N$.

So, we conclude that the system under consideration has the solution corresponding to irrep. Besides the reparametrization and scale invariance, this solution possesses an extra gauge freedom.

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