Cohomology of Currents of arbitrary Spin in AdS_3

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Abstract

We study conserved currents of any integer or half integer spin built from massless scalar and spinor fields in AdS_3. 2-forms dual to the conserved currents in AdS_3 are shown to be exact in the class of infinite expansions in higher derivatives of the matter fields with the coefficients containing inverse powers of the cosmological constant. This property has no analog in the flat space and may be related to the holography of the AdS spaces.

1 Introduction

AdS geometry plays important role in the theory of higher spin (HS) gauge fields (for a brief review see [1]) because interactions of HS gauge fields contain negative powers of the cosmological constant [2]. HS gauge fields in AdS_3 are not propagating in analogy with the usual Chern-Simons gravitational and Yang-Mills fields, although the HS gauge symmetries are still nontrivial. One can construct the HS currents from the matter fields of spin 0 and spin 1/2, which couplings to HS gauge potentials describe interactions of the matter via HS gauge fields.

Schematically, the equations of motion in the gauge field sector have a form $R = J(C;W)$, where $R = dW - W \wedge W$ denotes all spin $s \geq 1$ curvatures built from the HS potential W, while C denotes the matter fields (precise definitions are given in the sect. 2). To analyze the problem perturbatively, one fixes a vacuum solution W_0 that solves $R_0 = 0$, assuming that $W = W_0 + W_1$, while C starts from the first-order part. When gravity is included, as is the case in the HS gauge theories, W_0 is different from zero and describes background geometry. In the lowest nontrivial order one gets

$$R_1 \equiv D_0 W_1 = J_2(C^2)\,, \tag{1.1}$$

where D_0 is built from W_0 while the 2-form $J_2(C^2)$ dual to the 3d current vector field obeys the conservation law $D_0 J_2(C^2) = 0$ on the free equations of motion of the matter fields.

A nonlinear system of equations of motion describing HS gauge interactions for the spin 0 and spin 1/2 matter fields in AdS_3 in all orders in interactions has been formulated both for massless [3] and massive [4] matter fields. An interesting property of the proposed equations discovered in [4] is that there exists a mapping of the full nonlinear system to the free one. This mapping is a nonlinear field redefinition having a form of infinite power series in higher derivatives of the matter fields and is therefore generically nonlocal. The coefficients of such expansions contain inverse powers of the cosmological constant and therefore do not admit a flat limit. We call such expansions in higher derivatives pseudolocal. Comparison of the results of [4] with (1.1) implies that such a field redefinition exists in a nontrivial model if $J_2(C^2) = D_0 U(C^2)$, where U is some pseudolocal functional of the matter fields. The cohomological interpretation with D_0 as de Rahm differential is straightforward because $D_0^2 = R_0 = 0$. The fact that $J_2(C^2)$ is exact in the class of pseudolocal functionals has already been demonstrated for the spin 2 current in [4], where we have found a pseudolocal U for the stress tensor constructed from a massless scalar field. In this talk, we generalize this result to the currents of an arbitrary integer or half integer spin.

2 Higher Spin and Matter Fields in AdS_3

The 3d HS gauge fields are described [5, 3] by a spacetime 1-form $W = dx^\mu W_\mu(y,\psi|x)$ depending on the spacetime coordinates x^μ ($\mu = 0,1,2$), auxiliary commuting spinor variables y_α (indices $\alpha\,,\,\beta\,,\gamma\, = 1,2$ are lowered and raised by the symplectic form $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\epsilon_{12} = \epsilon^{12} = 1$, $A^\alpha = \epsilon^{\alpha\beta}A_\beta$, $A_\alpha = A^\beta\epsilon_{\beta\alpha}$), and the central involutive element ψ,

$$W_\mu(y,\psi|x) = \sum_{n=0}^{\infty} \frac{1}{2in!}\ \left[\omega_{\mu,\ \alpha(n)}(x) + \lambda\psi\ h_{\mu,\ \alpha(n)}(x)\right]\ y^{\alpha_1}\dots y^{\alpha_n}\,. \tag{2.1}$$

A constant parameter λ is to be identified with the inverse radius of AdS_3.

The HS gauge algebra is a Lie superalgebra built via (anti)commutators from the associative algebra spanned by the elements $a(y,\psi)$ with the product law "$*$" defined via the generating relations $y_\alpha * y_\beta - y_\beta * y_\alpha = 2i\epsilon_{\alpha\beta}$, $y_\alpha * \psi = \psi * y_\alpha$, $\psi * \psi = 1$. (The boson-fermion parity π is defined in a usual way as $a(-y,\psi) = (-)^{\pi(a)}\,a(y,\psi)$.) The field strength is [6, 5] $R(y,\psi|x) = dW(y,\psi|x) - W(y,\psi|x) * \wedge W(y,\psi|x)$. The role of the element ψ is to make the 3d HS superalgebra semisimple ($hs(2)\oplus hs(2)$, where, in notation of [6], $hs(2)$ is a superalgebra spanned by the ψ-independent elements $a(y)$), with simple components singled out by the projectors $P_\pm = \frac{1}{2}(1\pm\psi)$. This is similar to the AdS_3 isometry algebra $o(2,2)\sim sp(2)\oplus sp(2)$, which is identified with a subalgebra of $hs(2)\oplus hs(2)$ spanned by $L_{\alpha\beta} = \frac{1}{2i}\ y_\alpha y_\beta$ and $P_{\alpha\beta} = \frac{1}{2i}\ y_\alpha y_\beta\psi$. We therefore identify the $o(2,2)$ components of $W(y,\psi|x)$ (2.1) with the gravitational Lorentz connection 1-form $\omega^{\alpha\beta}(x) = dx^\mu\omega_{\mu,}{}^{\alpha\beta}(x)$ and the dreibein 1-form $h^{\alpha\beta}(x) = dx^\mu h_{\mu,}{}^{\alpha\beta}(x)$. Since AdS_3 algebra $o(2,2)$ is a proper subalgebra of the d3 HS algebra it is a consistent ansatz to require the vacuum value of $W(y,\psi|x)$ to be non-zero only in the spin 2 sector. Then the equation $R_0 = 0$ is equivalent to the $o(2,2)$ zero-curvature conditions $d\omega_{\alpha\beta} = \omega_{\alpha\gamma}\wedge\omega_\beta{}^\gamma + \lambda^2 h_{\alpha\gamma}\wedge h_\beta{}^\gamma$, $dh_{\alpha\beta} = \omega_{\alpha\gamma}\wedge h_\beta{}^\gamma + \omega_{\beta\gamma}\wedge h_\alpha{}^\gamma$. Supposing the dreibein $h_{\nu,}{}^{\alpha\beta}$ to be non-degenerate, one finds that these equations reduce to the zero-torsion condition and the equation $\mathcal{R}_{\alpha\beta} = -\lambda^2 h_{\alpha\gamma}\wedge h_\beta{}^\gamma$ for the Riemann tensor 2-form $\mathcal{R}_{\alpha\beta}$, thus describing AdS_3 with radius λ^{-1}.

The massless Klein-Gordon and Dirac equations in AdS_3 read

$$\Box C = \frac{3}{2}\lambda^2\ C \qquad \text{and} \qquad h^\mu{}_{,\alpha}{}^\beta\nabla_\mu C_\beta = 0 \tag{2.2}$$

for the spin 0 boson field $C(x)$ and spin $\frac{1}{2}$ fermion field $C_\alpha(x)$. Here $\Box = \nabla^\mu\nabla_\mu$, where ∇_μ is the full covariant derivative with the symmetric Christoffel connection defined via the metric postulate $\nabla_\mu h_{\nu,}{}^{\alpha\beta} = 0$.

The "unfolded" formulation [7] of the equations (2.2) in the form of some covariant constancy conditions is most convenient for the analysis of cohomology of currents. To this end one introduces an infinite set of symmetric multispinors $C_{\alpha_1\dots\alpha_n}$ for all $n \geq 0$. (We will assume total symmetrization of indices denoted by the same letter and will use the notation $C_{\alpha(n)} = C_{\alpha_1\dots\alpha_n}$ when only a number of indices is important.) As shown in [7], the infinite chain of equations

$$D^L C_{\alpha(n)} = \frac{i}{2}\left[h^{\beta\gamma}C_{\beta\gamma\alpha(n)} - \lambda^2 n(n-1)\ h_{\alpha\alpha}C_{\alpha(n-2)}\right]\,, \tag{2.3}$$

where D^L is the background Lorentz covariant differential, $D^L C_{\alpha(n)} = dC_{\alpha(n)} + n\ \omega_\alpha{}^\gamma C_{\gamma\alpha(n-1)}$, is equivalent to the equations (2.2) for the lowest rank components C and C_α along with some constraints expressing highest multispinors via highest spacetime derivatives of C and C_α. For example, for bosons

$$C_{\alpha(2n)}(x) = (-2i)^n\ h^{\nu_1}{}_{,\alpha\alpha}h^{\nu_2}{}_{,\alpha\alpha}\dots h^{\nu_n}{}_{,\alpha\alpha}\ \nabla_{\nu_1}\nabla_{\nu_2}\dots\nabla_{\nu_n}\ C(x)\,, \tag{2.4}$$

where ∇_μ is a full background derivative (for multispinors $\nabla_\mu C_{\alpha(n)} = D^L_\mu C_{\alpha(n)}$).

Following [7], let us introduce the generating function

$$C(y,\psi|x) = \sum_{n=0}^{\infty}\frac{1}{n!}\ (\lambda^{-1}\psi)^{[\frac{n}{2}]}\ C_{\alpha_1\dots\alpha_n}(x)\ y^{\alpha_1}\dots y^{\alpha_n} = \lambda^{\frac{1}{2}\pi(C)}\ \tilde{C}(\lambda^{-\frac{1}{2}}y,\psi|x)\,, \tag{2.5}$$

where $[n+a]=n$, $\forall n\in Z$ and $0\leq a<1$, and the boson-fermion parity $\pi(C)=0(1)$ for even (odd) functions $C(y)$. The equations (2.3) can be rewritten in the form [7],

$$D^L C(y,\psi)=\frac{i\lambda}{2}\,\psi\, h^{\alpha\beta}\left[\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial y^\beta}-y_\alpha y_\beta\right]\,C(y,\psi)\,,\qquad D^L=d-\omega^{\alpha\beta}y_\alpha\frac{\partial}{\partial y^\beta}\,. \tag{2.6}$$

The fields $C_{\alpha_1\dots\alpha_n}$ are identified with all on-mass-shell nontrivial derivatives of the matter fields according to (2.4). The condition that the system is on–mass–shell is encoded in the fact that the multispinors $C_{\alpha_1\dots\alpha_n}$ are totally symmetric. This allows us to work with $C_{\alpha_1\dots\alpha_n}$ instead of explicit derivatives of the matter fields. Consider a function $F[C_{\alpha(n)}(x)]$ of all components of $C_{\alpha_1\dots\alpha_n}(x)$ at some fixed point x. F is not supposed to contain any derivatives with respect to the spacetime coordinates x and therefore looks like a local function of matter fields. One has to be careful however because, when the equations (2.3) hold, (2.4) is true. We will therefore call a function $F[C_{\alpha(n)}]$ pseudolocal if it is an infinite expansion in the field variables $C_{\alpha(n)}(x)$ and local if F is a polynomial. In terms of the generating functions $C(y,\psi|x)$ this can be reformulated as follows. Let $F(C|x)$ be some functional of the generating function $C(y,\psi|x)$ at some fixed point of spacetime x. According to (2.4) its spacetime locality is equivalent on–mass–shell to the locality in the y space. Indeed, from (2.6) it follows that the derivatives in the spinor variables form in a certain sense a square root of the spacetime derivatives.

The equation (1.1) for the d3 HS system reads (in the rest of the paper we use the symbol D instead of D_0)

$$DW_1(y,\psi|x)=J(C^2)(y,\psi|x) \tag{2.7}$$

with the background AdS covariant differential

$$D=D^L-\lambda\psi\, h^{\alpha\beta}\, y_\alpha\frac{\partial}{\partial y^\beta}=d-(\omega^{\alpha\beta}+\lambda\psi\, h^{\alpha\beta})\, y_\alpha\frac{\partial}{\partial y^\beta}\,. \tag{2.8}$$

Since $D^2=R_0=0$ our problem is to study the cohomology of D (2.8). Clearly, D commutes with the Euler operator $N=y^\alpha\frac{\partial}{\partial y^\alpha}$ which eigenvalues are identified with spin s via $N=2(s-1)$. The problem therefore is to be analyzed for different spins independently.

Conserved currents of an arbitrary integer spin in d4 Minkowski spacetime were considered in [8]. For $d=2$, HS conserved currents were constructed in [9].

In the case of AdS_3, conserved currents of any integer spin $s\geq 1$ built from two massless scalar fields C, C' have a form

$$\begin{aligned}J^{(s)}_{\mu,\,\alpha(2s-2)}(C,C') \;&=\; \sum_{k=0}^{s-2}\frac{2(-1)^k}{(2k+1)!(2s-2k-3)!}\,h_{\mu,}{}^{\gamma\gamma}\,C_{\gamma\alpha(2k+1)}C'_{\gamma\alpha(2s-2k-3)}\\ &+\sum_{k=0}^{s-1}\frac{(-1)^k}{(2k)!(2s-2k-2)!}\,h_{\mu,}{}^{\gamma\gamma}\,\left[C_{\gamma\gamma\alpha(2k)}C'_{\alpha(2s-2k-2)}-C_{\alpha(2k)}C'_{\gamma\gamma\alpha(2s-2k-2)}\right]\,.\end{aligned} \tag{2.9}$$

Analogously, one can write down the currents of any integer spin $s\geq 1$ built from two massless spinors, and the "supercurrents" of any half integer spin $s\geq 3/2$ built from one scalar and one spinor [10]. The lowest spin conserved currents read

$$J^{(1)}_\mu(C,C')=h_{\mu,}{}^{\gamma\gamma}\,(C_{\gamma\gamma}C'-CC'_{\gamma\gamma})\,,\qquad J^{(1)}_\mu(C_\alpha,C'_\alpha)=h_{\mu,}{}^{\gamma\gamma}\,C_\gamma C'_\gamma\,, \tag{2.10}$$

$$J^{(3/2)}_{\mu,\,\alpha}(C,C'_\alpha)=h_{\mu,}{}^{\gamma\gamma}\,(C_{\gamma\gamma}C'_\alpha-CC'_{\gamma\gamma\alpha}+2C_{\gamma\alpha}C'_\gamma)\,, \tag{2.11}$$

$$J^{(2)}_{\mu,\,\alpha\alpha}(C,C')=\frac{1}{2}\,h_{\mu,}{}^{\gamma\gamma}\,(C_{\gamma\gamma}C'_{\alpha\alpha}-CC'_{\gamma\gamma\alpha\alpha}-C_{\gamma\gamma\alpha\alpha}C'+C_{\alpha\alpha}C'_{\gamma\gamma}+4\,C_{\gamma\alpha}C'_{\gamma\alpha})\,. \tag{2.12}$$

These currents are all local, containing a finite number of terms (i.e., higher derivatives (2.4)). The same expressions remain valid in the flat limit with $\nabla_\mu\to\partial_\mu$ in (2.4).

3 Generating Functions

To analyze the cohomology problem for currents of an arbitrary spin we first elaborate a technique operating with the generating functions (2.5) rather than with the individual multispinors. A generic

Lorentz covariant spacetime 1-form of spin $s = n/2 + 1$ bilinear in two different matter fields C and C' and their on–mass–shell nontrivial derivatives is

$$\begin{aligned}
\Phi_{\alpha(n)}\,(C,C'|x) \;&=\; \sum_{k+l=n-2}\sum_{m=0}^{\infty} a^{klm}\; h_{\alpha\alpha} C_{\alpha(k)}{}^{\beta(m)}(x)\; C'_{\alpha(l)\beta(m)}(x) \\
&+ \sum_{k+l=n-1}\sum_{m=0}^{\infty}\Big[b_1^{klm}\; h_\alpha{}^\gamma C_{\gamma\alpha(k)}{}^{\beta(m)}(x)\; C'_{\alpha(l)\beta(m)}(x) + b_2^{klm}\; h_\alpha{}^\gamma C_{\alpha(k)}{}^{\beta(m)}(x)\; C'_{\gamma\alpha(l)\beta(m)}(x)\Big] \\
&+ \sum_{k+l=n}\sum_{m=0}^{\infty}\Big[e_1^{klm}\; h^{\gamma\gamma} C_{\gamma\gamma\alpha(k)}{}^{\beta(m)}(x)\; C'_{\alpha(l)\beta(m)}(x) + e_2^{klm}\; h^{\gamma\gamma} C_{\gamma\alpha(k)}{}^{\beta(m)}(x)\; C'_{\gamma\alpha(l)\beta(m)}(x) \\
&+ e_3^{klm}\; h^{\gamma\gamma} C_{\alpha(k)}{}^{\beta(m)}(x)\; C'_{\gamma\gamma\alpha(l)\beta(m)}(x)\Big]\,, \qquad (3.1)
\end{aligned}$$

where a^{klm}, $b_{1,2}^{klm}$, and $e_{1,2,3}^{klm}$ are arbitrary constants and $h_{\alpha\alpha}$ is the dreibein 1-form. Introducing $\Phi(y,\psi|x) = \Phi_{\alpha_1\dots\alpha_n}(\psi|x)\, y^{\alpha_1}\dots y^{\alpha_n}$, one can equivalently rewrite this formula as

$$\begin{aligned}
\Phi(y,\psi|x) \;&=\; h_{\alpha\alpha}\,\frac{1}{(2\pi)^2}\oint dr\oint ds\oint \tau^{-2}\,d\tau\int d^2u\,d^2v\;\exp\left\{\frac{i}{\tau}(u_\gamma v^\gamma)\right\} \\
&\times C(u-ry,\psi|x)\; C'(v+sy,\psi|x)\,[f_1(r,s,\tau)\,y^\alpha y^\alpha + f_2(r,s,\tau)\,y^\alpha u^\alpha + f_3(r,s,\tau)\,y^\alpha v^\alpha \\
&+ f_4(r,s,\tau)\,u^\alpha u^\alpha + f_5(r,s,\tau)\,u^\alpha v^\alpha + f_6(r,s,\tau)\,v^\alpha v^\alpha]\,. \qquad (3.2)
\end{aligned}$$

Here r, s, and τ are complex variables, u_α and v_α ($\alpha = 1,2$) are spinor variables. The quantities $f_i(r,s,\tau)$, $i = 1,\dots,6$ are polynomials in r^{-1} and s^{-1} and formal series in τ^{-1},

$$f_i(r,s,\tau) = \sum_{\substack{0<k,\,l<p,\\ p<\infty}}\sum_{m=1}^{\infty} f_i(k,l,m)\; r^{-k}s^{-l}\tau^{-m}\,. \qquad (3.3)$$

The contour integrations are normalized as $\oint \tau^{-n}d\tau = \delta_n^1$. The Gaussian integrations with respect to u_α and v_α should be completed prior the contour integrations.

Inserting (2.5) into (3.2) and completing elementary integrations one arrives at (3.1) with the coefficients a^{klm}, $b_{1,2}^{klm}$, and $e_{1,2,3}^{klm}$ expressed via $f_i(k,l,m)$. Therefore (3.2) indeed describes a general Lorentz covariant 1-form bilinear in the matter fields. Note that the formula (3.2) produces a spacetime local expression if all the coefficients f_i contain a finite number of terms in (3.3) and pseudolocal if some of the expansions in negative powers of τ are infinite.

In practice, the following representations of rank $n = 0,1,2,3$ differential forms $\Phi_n(x)$ are most convenient,

$$\begin{aligned}
\Phi_{0,3}(y,\psi|x) \;&=\; H_{0,3}\,\frac{1}{(2\pi)^2}\oint\frac{dz}{z}\oint\frac{d\bar z}{\bar z}\oint\frac{d\tau}{\tau^2}\int d^2q\,d^2\bar q\;\exp\left\{-\frac{1}{2\tau}(q_\gamma\bar q^\gamma)\right\} \\
\times C\left[\frac{1}{2}(q+\bar q) - \frac{1}{2i}(z-\bar z)y\,,\;\psi\;\middle|\;x\right] C'\left[\frac{1}{2i}(q-\bar q) + \frac{1}{2}(z+\bar z)y\,,\;\psi\;\middle|\;x\right] E_{0,3}(z,\bar z,\tau)\,, \qquad (3.4) \\
\Phi_{1,2}(y,\psi|x) \;&=\; H_{1,2\;\alpha\alpha}\,\frac{1}{(2\pi)^2}\oint dz\oint d\bar z\oint\tau^{-2}\,d\tau\int d^2q\,d^2\bar q\;\exp\left\{-\frac{1}{2\tau}(q_\gamma\bar q^\gamma)\right\} \\
&\times C\left[\frac{1}{2}(q+\bar q) - \frac{1}{2i}(z-\bar z)y\,,\;\psi\;\middle|\;x\right] C'\left[\frac{1}{2i}(q-\bar q) + \frac{1}{2}(z+\bar z)y\,,\;\psi\;\middle|\;x\right] \\
&\times\Big\{R_{1,2}(z,\bar z,\tau)\,y^\alpha y^\alpha + \frac{1}{2\tau\bar z}W_{1,2}(z,\bar z,\tau)\,y^\alpha\bar q^\alpha + \frac{1}{2\tau z}\bar W_{1,2}(z,\bar z,\tau)\,y^\alpha q^\alpha \\
&+ \frac{1}{2\tau^2\bar z^2}Y_{1,2}(z,\bar z,\tau)\,\bar q^\alpha\bar q^\alpha + \frac{1}{2\tau^2 z^2}\bar Y_{1,2}(z,\bar z,\tau)\,q^\alpha q^\alpha + \frac{1}{2\tau^2 z\bar z}V_{1,2}(z,\bar z,\tau)\,q^\alpha\bar q^\alpha\Big\}\,. \qquad (3.5)
\end{aligned}$$

Here $H_0 = \psi$, $H_3 = -\frac{\lambda^2}{12}\,h_{\alpha\beta}\wedge h^\beta{}_\gamma\wedge h^{\gamma\alpha}$, $H_{1\;\alpha\alpha} = h_{\alpha\alpha}$, and $H_{2\;\alpha\alpha} = -\frac{\lambda}{2}\psi\,h_{\alpha\beta}\wedge h^\beta{}_\alpha$, where the factors of ψ, $-\frac{\lambda}{2}\psi$, and $-\frac{\lambda^2}{12}$ are introduced for future convenience. It is not hard to see that the expressions (3.4)

and (3.5) reproduce arbitrary Lorentz covariant forms bilinear in the matter fields and their on-mass-shell nontrivial derivatives.

Let n, $\bar{n}$, and n_τ be the following operators,

$$n = z\,\frac{\partial}{\partial z}\,, \qquad \bar{n} = \bar{z}\,\frac{\partial}{\partial \bar{z}}\,, \qquad n_\tau = \tau\,\frac{\partial}{\partial \tau} \tag{3.6}$$

(using the same notations for their eigenvalues). The quantities $R_{1,2}(z,\bar{z},\tau)$, $W_{1,2}(z,\bar{z},\tau)$, $\bar{W}_{1,2}(z,\bar{z},\tau)$, $Y_{1,2}(z,\bar{z},\tau)$, $\bar{Y}_{1,2}(z,\bar{z},\tau)$, $V_{1,2}(z,\bar{z},\tau)$, and $E_{0,3}(z,\bar{z},\tau)$ give a non-zero contribution to (3.4) and (3.5) when n, $\bar{n}$, and n_τ satisfy the following restrictions:

R_1, R_2	$n \leq -1$	$\bar{n} \leq -1$	$n_\tau \leq -1$
W_1, W_2	$n \leq -1$	$\bar{n} \leq 0$	$n_\tau \leq -1$
$\bar{W}_1$, $\bar{W}_2$	$n \leq 0$	$\bar{n} \leq -1$	$n_\tau \leq -1$
Y_1, Y_2	$n \leq -1$	$\bar{n} \leq 1$	$n_\tau \leq -1$
$\bar{Y}_1$, $\bar{Y}_2$	$n \leq 1$	$\bar{n} \leq -1$	$n_\tau \leq -1$
V_1, V_2, E_0, E_3	$n \leq 0$	$\bar{n} \leq 0$	$n_\tau \leq -1$

(3.7)

Beyond these regions, the coefficients do not contribute and therefore their values can be fixed arbitrarily. As a result, the quantities $R_{1,2}$, $W_{1,2}$, $\bar{W}_{1,2}$, ... are defined modulo arbitrary polynomials $P(\tau) = \sum_{k=0}^{k_0} P_k \tau^k$.

For the two-component spinors, antisymmetrization over any three two-component spinor indices gives zero. This is expressed by the identity $a_\alpha(b_\beta c^\beta) + b_\alpha(c_\beta a^\beta) + c_\alpha(a_\beta b^\beta) = 0$ valid for any three commuting two-component spinors a_α, b_α, and c_α. As a result, the forms discussed so far are not all independent. The ambiguity in adding any terms which vanish as a consequence of this identity can be expressed in a form of some equivalence (gauge) transformations of the coefficients in (3.2) and (3.5). We call these equivalence transformations Fierz transformations. Using the partial integrations w.r.t. τ, z, and $\bar{z}$, one can check that the transformations

$$\begin{aligned}
&\delta R_{1,2} = -\partial_\tau \chi_{1,2}\,, \qquad \delta W_{1,2} = -\partial_\tau \xi_{1,2} + 2i\bar{n}\chi_{1,2}\,, \qquad \delta \bar{W}_{1,2} = -\partial_\tau \bar{\xi}_{1,2} - 2in\chi_{1,2}\,,\\
&\delta V_{1,2} = -i(n\xi_{1,2} - \bar{n}\bar{\xi}_{1,2})\,, \qquad \delta Y_{1,2} = i(\bar{n}-1)\xi_{1,2}\,, \qquad \delta \bar{Y}_{1,2} = -i(n-1)\bar{\xi}_{1,2}
\end{aligned} \tag{3.8}$$

with arbitrary parameters $\chi_{1,2}(z,\bar{z},\tau)$, $\xi_{1,2}(z,\bar{z},\tau)$, and $\bar{\xi}_{1,2}(z,\bar{z},\tau)$ describe all possible Fierz transformations of the 1- and 2-forms (3.5).

4 On-Mass-Shell Current Complex

In this section we study the on–mass-shell action of the operator D (2.8) on the differential forms defined in sect. 3. The advantage of the formulation of the dynamical equations in the unfolded form (2.6) is that it expresses the spacetime derivative of C via some operators acting in the auxiliary spinor space. As a result, on–mass–shell action of D reduces to some mapping $\mathcal{D}$ acting on the coefficients in the formulae (3.4)-(3.5).

Let us consider the example of a 0-form. Using the Leibnitz rule for D^L and taking into account the equations of motion (2.6) and the zero torsion condition $D^L h_{\alpha\alpha} = 0$, one arrives at the 1-form $\hat{\Phi}_1 = D\Phi_0$ with the coefficients $R_1^{\mathcal{D}}(E_0)$, $W_1^{\mathcal{D}}(E_0)$, $\bar{W}_1^{\mathcal{D}}(E_0)$, $Y_1^{\mathcal{D}}(E_0)$, $\bar{Y}_1^{\mathcal{D}}(E_0)$, and $V_1^{\mathcal{D}}(E_0)$ of the form

$$\begin{aligned}
&R_1^{\mathcal{D}}(E_0) = -\frac{i\lambda}{2}E_0\,, \qquad W_1^{\mathcal{D}}(E_0) = i\lambda(1-i\tau)E_0\,, \qquad \bar{W}_1^{\mathcal{D}}(E_0) = i\lambda(1+i\tau)E_0\,,\\
&V_1^{\mathcal{D}}(E_0) = -i\lambda(1+\tau^2)E_0\,, \qquad Y_1^{\mathcal{D}}(E_0) = \bar{Y}_1^{\mathcal{D}}(E_0) = 0\,.
\end{aligned} \tag{4.1}$$

Analogously one derives the mapping $\mathcal{D}:\ \Phi_i(y) \to \hat{\Phi}_{i+1}(y) = D\Phi_i(y)\Big|_{on-shell}$, $i = 1,2$ on the coefficients of the differential forms (3.4), (3.5), $\mathcal{D}\ \{R_1, W_1, ...\} = \{R_2^{\mathcal{D}}, W_2^{\mathcal{D}}, ...\}$, $\mathcal{D}\ \{R_2, W_2, ...\} = E_3^{\mathcal{D}}$ with

$$R_2^{\mathcal{D}} \;=\; -(1-i\tau)nR_1 - (1+i\tau)\bar{n}R_1 + 2R_1 + \frac{i}{4}(1+i\tau)\partial_\tau W_1 - \frac{i}{4}(1-i\tau)\partial_\tau \bar{W}_1$$

$$+\frac{1}{4}(W_1+\bar W_1)-\frac{1}{2}(nW_1+\bar n\bar W_1)\,, \tag{4.2}$$

$$\begin{aligned} W_2^{\mathcal D} &= -\frac{i}{2}\partial_\tau[(1+\tau^2)W_1]+\frac{3}{2}(1+i\tau)W_1+2(1+\tau^2)\bar nR_1+\frac{1}{2}(1-i\tau)\bar n\bar W_1+(1-2n)Y_1\\ &\quad -\frac{1}{2}(1+i\tau)(\bar n-1)W_1+i(1+i\tau)\partial_\tau Y_1+\left(\frac{3}{2}-\bar n\right)V_1-\frac{i}{2}(1-i\tau)\partial_\tau V_1\,, \end{aligned} \tag{4.3}$$

$$\begin{aligned} V_2^{\mathcal D} &= \frac{1}{2}(1+\tau^2)(nW_1+\bar n\bar W_1)+\frac{1}{2}(1+i\tau)(\bar n-1)V_1+\frac{1}{2}(1-i\tau)(n-1)V_1+V_1\\ &\quad +(1+i\tau)nY_1+(1-i\tau)\bar n\bar Y_1\,, \end{aligned} \tag{4.4}$$

$$\begin{aligned} Y_2^{\mathcal D} &= \frac{1}{2}(1+\tau^2)(\bar n-1)W_1-i(1+\tau^2)\partial_\tau Y_1+(1+i\tau)Y_1+(1-i\tau)nY_1\\ &\quad +\frac{1}{2}(1-i\tau)(\bar n-1)V_1\,, \end{aligned} \tag{4.5}$$

$$\begin{aligned} E_3^{\mathcal D} &= 4in\bar n(1+\tau^2)R_2+in\bar n(W_2+\bar W_2)-3\tau(nW_2-\bar n\bar W_2)-\tau n\bar n(W_2-\bar W_2)\\ &\quad +\partial_\tau[(1+\tau^2)(nW_2-\bar n\bar W_2)]+2\partial_\tau(nY_2-\bar n\bar Y_2)-(n-\bar n)\,\partial_\tau V_2\\ &\quad +2i(n\bar n-n-\bar n+1)V_2+i(n+\bar n-2)\,\tau\partial_\tau V_2+i(1+\tau^2)\,\partial_\tau\partial_\tau V_2\\ &\quad -3i(nW_2+\bar n\bar W_2)+(i+\tau)\,n(n+1)W_2+(i-\tau)\,\bar n(\bar n+1)\bar W_2\\ &\quad -4i(nY_2+\bar n\bar Y_2)+2i\,\tau\partial_\tau(nY_2+\bar n\bar Y_2)+2i[n(n+1)Y_2+\bar n(\bar n+1)\bar Y_2]\,. \end{aligned} \tag{4.6}$$

The corresponding formulae for the parameters $\bar W_2^{\mathcal D}$ and $\bar Y_2^{\mathcal D}$ are given by (4.3) and (4.5) respectively with the replacements $i\to -i$, $n\leftrightarrow\bar n$, $W_1\leftrightarrow\bar W_1$, and $Y_1\leftrightarrow\bar Y_1$ on r.h.s.-s. As expected, $\mathcal D^2=0$ and therefore the mapping $\mathcal D$ defines a complex $(T,\mathcal D)$ with $T=\bigoplus\limits_{i=0,1,2,3}T_i$, where $T_{0,3}=\{E_{0,3}\}$ and $T_{1,2}=\{R_{1,2},W_{1,2},\bar W_{1,2},V_{1,2},Y_{1,2},\bar Y_{1,2}\}$. The reformulation of the problem in terms of $(T,\mathcal D)$ effectively accounts the fact that the fields are on–mass–shell. We identify the cohomology of currents with the cohomology of $\mathcal D$.

The remarkable property of the mapping $\mathcal D$ is that it contains z, $\bar z$, $\frac{\partial}{\partial z}$, and $\frac{\partial}{\partial\bar z}$ only via n and $\bar n$ (3.6), thus implying the separation of variables: the differential $\mathcal D$ leaves invariant eigensubspaces of n and $\bar n$. In fact, this is the main reason for using the particular representation (3.4)-(3.5).

As expected, the system (4.2)-(4.5) is consistent with the Fierz transformations (3.8). Namely, any Fierz transformation of the quantities R_1, W_1, ... leads to some Fierz transformation of the quantities $R_2^{\mathcal D}$, $W_2^{\mathcal D}$, ... , and any Fierz transformation of R_2, W_2, ... does not affect the parameter $E_3^{\mathcal D}$ (4.6).

Following [8] we study the currents containing the minimal possible number of spacetime derivatives for a given spin s. From (2.4) it is clear that this is the case if the number of the contracted indices β in (3.1) is zero. Since the number of contractions is $-(n_\tau+1)$ (see sect. 3) we consider 2-forms $\Phi_2^{n,\bar n}$ with $n_\tau=-1$. Thus we set in (3.5)

$$\begin{aligned} &R_2=\alpha_R(n,\bar n)\;z^n\bar z^{\bar n}\tau^{-1}\,,\quad W_2=\alpha_W(n,\bar n)\;z^n\bar z^{\bar n}\tau^{-1}\,,\quad \bar W_2=\alpha_{\bar W}(n,\bar n)\;z^n\bar z^{\bar n}\tau^{-1}\,,\\ &Y_2=\alpha_Y(n,\bar n)\;z^n\bar z^{\bar n}\tau^{-1}\,,\quad \bar Y_2=\alpha_{\bar Y}(n,\bar n)\;z^n\bar z^{\bar n}\tau^{-1}\,,\quad V_2=\alpha_V(n,\bar n)\;z^n\bar z^{\bar n}\tau^{-1} \end{aligned} \tag{4.7}$$

with some constant parameters $\alpha_R(n,\bar n)$, $\alpha_W(n,\bar n)$, ... $\sim\lambda^{[s]}$, where $s=1-\frac{1}{2}(n+\bar n)$. The conservation condition means that $\Phi_2^{n,\bar n}$ should be $\mathcal D$-closed. The requirement $E_3^{\mathcal D}=0$ modulo terms that do not contribute to (3.4) imposes the following conditions

$$\begin{aligned} &4n\bar n\alpha_R+(n+\bar n-2)(n\alpha_W+\bar n\alpha_{\bar W})+2n(n-2)\alpha_Y+2\bar n(\bar n-2)\alpha_{\bar Y}=0\,,\\ &n\alpha_W-\bar n\alpha_{\bar W}+2(n\alpha_Y-\bar n\alpha_{\bar Y})=0\,,\qquad \alpha_V=0\,, \end{aligned} \tag{4.8}$$

for $n\neq 1$, $\bar n\neq 1$. For $n=1$ or $\bar n=1$ $\Phi_2^{n,\bar n}$ is closed as a consequence of (3.7).

Our problem is to investigate whether there exist coefficients R_1, W_1, ... such that $R_2^{\mathcal D}$, $W_2^{\mathcal D}$, ... have a form (4.7). To this end one has to solve the system (4.2)-(4.5) in terms of the formal series $f(\tau)=\sum_{k=-\infty}^{p<\infty}f_k\tau^k$.

The Fierz transformations (3.8) for Φ_1 together the exact shifts of R_1, W_1, ... by any $R_1^{\mathcal D}$, $W_1^{\mathcal D}$, ... (4.1) produce the following equivalence transformations

$$\delta R_1=-\partial_\tau\chi_1+\varepsilon\,,\qquad \delta W_1=-\partial_\tau\xi_1+2i\bar n\chi_1-2(1-i\tau)\varepsilon\,,$$

$$\delta\bar{W}_1 = -\partial_\tau\bar{\xi}_1 - 2in\chi_1 - 2(1+i\tau)\varepsilon\,, \qquad \delta V_1 = 2(1+\tau^2)\varepsilon - i(n\xi_1 - \bar{n}\bar{\xi}_1)\,,$$
$$\delta Y_1 = i(\bar{n}-1)\xi_1\,, \qquad \delta\bar{Y}_1 = -i(n-1)\bar{\xi}_1\,. \tag{4.9}$$

We consider separately two cases: (i) with $n = 1$, $\bar{n} = 1-2s$ or $\bar{n} = 1$, $n = 1-2s$ and (ii) with $n < 1$ and $\bar{n} < 1$. As shown below, the case (i) corresponds to the nontrivial physical conserved currents, whereas the case (ii) describes all possible "improvements".

Let us start with the case (i) setting for definiteness $\bar{n} = 1$. According to (3.7), Y_2 is the only coefficient giving a non-zero contribution to $\Phi_2^{n,1}(y)$. Obviously, a 2-form with $\bar{n} = 1$ is invariant under the transformations (3.8). The only non-trivial equation is (4.5). With $Y_2^{\mathcal{D}}$ (4.7) it takes the form

$$(1+\tau^2)\,\partial_\tau Y_1 = -i(1+i\tau)\,Y_1 + i(2s-1)(1-i\tau)\,Y_1 + i\alpha_Y(1-2s,1)\,z^{1-2s}\bar{z}\,\tau^{-1} \tag{4.10}$$

modulo terms polynomial in τ. It is not hard to see that a generic solution of (4.10) is

$$Y_1(z,\bar{z},\tau) = -\frac{i}{2}\,\alpha_Y(1-2s,1)z^{1-2s}\bar{z}\,(1-i\tau)(1+i\tau)^{2s-1}\ln(1+\tau^{-2})$$
$$+\,\sigma\,z^{1-2s}\bar{z}\,(1-i\tau)(1+i\tau)^{2s-1}\,\ln\left(\frac{1+i\tau^{-1}}{1-i\tau^{-1}}\right) + Q(\tau)\,, \tag{4.11}$$

where σ is an arbitrary constant and $Q(\tau)$ is some inessential polynomial. The logarithms are treated as power series in τ^{-1}.

At any σ, the solution (4.11) is an infinite series in τ^{-1}, thus corresponding to some pseudolocal 1-form. Thus, the 2-forms $\Phi_2^s(y|x)$ constructed with the polynomials Y_2 at $\bar{n} = 1$ and with $\bar{Y}_2$ at $n = 1$ are $\mathcal{D}$-closed and cannot be represented as $D\Phi_1^s(y|x)$ with a spacetime local $\Phi_1^s(y|x)$. We therefore argue that the 2-form $\Phi_2^s(y)$ describes a physical conserved current of spin s. The currents (2.9) as well as the currents containing fermions are reproduced via Y_2 (4.7) with $\alpha_Y(1-2s,1) = 2^{2s-1}\,(\lambda\psi)^{[s]}$. The formula (4.11) solves the problem of reformulation of the physical currents as pseudolocally exact 2-forms. Note that the currents generated by $Y_2(n = 1-2s, \bar{n} = 1)$ and $\bar{Y}_2(n = 1, \bar{n} = 1-2s)$ are equivalent by the interchange $C \leftrightarrow C'$.

Note that the solution (4.11) is not unique, containing an arbitrary parameter σ. Since the transformations (4.9) are trivial for Y_1 at $\bar{n} = 1$, this one-parametric ambiguity cannot be compensated this way. This means that we have found a pseudolocal 1-form that is $\mathcal{D}$-closed but not $\mathcal{D}$-exact, i.e., the cohomology group $H^1(T,\mathcal{D})$ is nontrivial. This fact is in agreement with the one-parametric ambiguity found in [4] for the spin 2 case.

Let us now consider the case of $n < 1$, $\bar{n} < 1$. Substituting (4.7) into the system (4.2)-(4.5) one can see that, if the conditions (4.8) guaranteeing that Φ_2 is $\mathcal{D}$-closed are satisfied, then it admits the following solution,

$$R_1(z,\bar{z},\tau) = \frac{1}{4n\bar{n}}\left(n\alpha_W + \bar{n}\alpha_{\bar{W}} - \frac{2n}{\bar{n}-1}\,\alpha_Y - \frac{2\bar{n}}{n-1}\,\alpha_{\bar{Y}}\right) z^n\bar{z}^{\bar{n}}\tau^{-1}\,, \quad n,\bar{n} < 0\,, \tag{4.12}$$

$$W_1(z,\bar{z},\tau) = \frac{\alpha_Y}{\bar{n}-1}\,z^n\bar{z}^{\bar{n}}\tau^{-1}\,, \quad \bar{W}_1(z,\bar{z},\tau) = \frac{\alpha_{\bar{Y}}}{n-1}\,z^n\bar{z}^{\bar{n}}\tau^{-1}\,, \quad V_1 = Y_1 = \bar{Y}_1 = 0\,. \tag{4.13}$$

We observe that the 1-form $\Phi_1^{n,\bar{n}}(y)$ leads to a spacetime local expression since R_1, W_1, and $\bar{W}_1$ (4.12), (4.13) are linear in τ^{-1}. Therefore, $\hat{\Phi}_2^{n,\bar{n}}(y|x) = D\Phi_1^{n,\bar{n}}(y|x)$ with some local $\Phi_1^{n,\bar{n}}(y|x)$. Thus, it is an "improvement" of the physical current 2-form $J(C^2)$ on the r.h.s. of (2.7), which can be compensated by a local field redefinition of the (HS) gauge fields.

To investigate what happens in the flat limit $\lambda \to 0$ one should use the generating function $\tilde{C}(y)$ (2.5) instead of $C(y)$. A simple analysis show that in terms of $\tilde{C}(y)$ the solution (4.11) contain an inverse power of λ together with each power of τ^{-1}. Hence, a representation of physical current 2-forms $\Phi_2^s(y)$ as some differentials $D\Phi_1^s(y)$ becomes meaningless in the flat limit.

Conclusion

It is shown that local conserved currents of an arbitrary spin in AdS_3 can be treated as "improvements" within the class of infinite power expansions in higher derivatives, i.e., 2-forms J dual to the physical

conserved currents are shown to be exact in this class, $J = DU$. The 1-forms U are constructed explicitly what allows us to write down nonlocal field redefinitions compensating matter sources in the equations of motion for the Chern-Simons gauge fields of all spins. The coefficients in the expansion of U in derivatives of the matter fields contain negative powers of the cosmological constant (i.e. positive powers of the AdS radius) and therefore do not admit a flat limit. The existence of U may be related to the holography in the AdS/CFT correspondence [11] since it indicates that local current interactions in AdS_3 are in a certain sense trivial and can, up to some surface terms, be compensated by a field redefinition. We expect similar phenomenon to take place for AdS_d with any d.

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References

[1] M. A. Vasiliev, *Int. J. Mod. Phys.* **D5** (1996) 763.

[2] E. S. Fradkin and M. A. Vasiliev, *Phys. Lett.* **B189** (1987) 89; *Nucl. Phys.* **B291** (1987) 141.

[3] M. A. Vasiliev, *Mod. Phys. Lett.* **A7** (1992) 3689.

[4] S. F. Prokushkin and M. A. Vasiliev, *Nucl. Phys.* **B545** (1999) 385, hep-th/9806236.

[5] M. P. Blencowe, *Class. Quantum Grav.* **6** (1989) 443.

[6] M. A. Vasiliev, *Fortschr. Phys.* **36** (1988) 33.

[7] M.A. Vasiliev, *Class. Quant. Grav.* **11** (1994) 649.

[8] F. A. Berends, G. J. H. Burgers, and H. van Dam, *Nucl. Phys.* **B271** (1986) 429.

[9] A. K. H. Bengtsson and I. Bengtsson, *Phys. Lett.* **B174** (1986) 294.

[10] S. F. Prokushkin and M. A. Vasiliev, hep-th/9907020, to be published in *Teor. Mat. Fiz.*

[11] J. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity", *Adv. Theor. Math. Phys* 2 (1998) 231, hep-th/9711200;
E. Witten, "Anti De Sitter Space and Holography", *Adv. Theor. Math. Phys.* 2 (1998) 253, hep-th/9802150.