Tree Form-Factors in N=4 Supersymmetric Yang-Mills

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Abstract

The construction of *perturbiner* (the solution of field equations which is a generating function for tree form-factors) in $N = 3$ ($N = 4$) supersymmetric Yang-Mills theory is reviewed.

1 Introduction

In this my talk I review the construction of perturbiner [3] in N=4 supersymmetric Yang-Mills.

To define the perturbiner, one first fixes a solution of linearized field equations (which are assumed to describe asymptotic one-particle states) of the type

$$\phi^{(1)} = \sum_{J=1}^{L} a_J \epsilon_J t_J e^{k_J x} = \sum_{J}^{L} \epsilon_J \hat{\mathcal{E}} \tag{1}$$

where x stands for a space-time coordinate, k_J stands for a momentum of the J-th particle, ϵ_J stands for a polarization of the J-th particle, t_J stands for a "polarization" of the J-th particle in the internal space (e.g. t_J is a generator of the color group), a_J is a symbol of annihilation/creation operator,

$$\hat{\mathcal{E}}_J = t_J \mathcal{E}, \; \mathcal{E} = a_J e^{k_J x} \tag{2}$$

The perturbiner is a complex solution of the field equations which is a formal power series in the "harmonics" $\mathcal{E}_J, J = 1, \ldots, L$ Eq. (2), the first-order term of the series being a solution of Eq.(1).

Notice that x-dependence of the perturbiner comes only via monomials in $\mathcal{E}_J, J = 1, \ldots, L$, on which differential operators entering the field equations act as algebraic operators, and existence/uniqueness [1] of the perturbiner normally takes place provided the set of momenta, $k_J, J = 1, \ldots, L$, is non-resonant, that is, provided none of linear combinations of $k_J, J = 1, \ldots, L$ with positive integer coefficients including more than one momentum gets to the mass shell. The physical meaning of the perturbiner is that its expansion in powers of symbols $a_J, J = 1, \ldots, L$ generates tree form-factors in the theory,

$$\phi^{ptb}(x, \{k\}, \{a\}) = \sum_{l=1}^{L} \sum_{\{J\}} a_{J_1} \ldots a_{J_l} < k_{J_1}, \ldots, k_{J_l} | \phi(x) | 0 >_{tree} \tag{3}$$

The form-factors for sufficiently high number of external particles are nontrivial objects even in the tree approximation, so the perturbiner can be viewed as an alternative way to describe them. We also believe that it is interesting by itself.

Notice that all the external mass-shell particles are arbitrarily considered as the out- ones. In principle, those with negative frequency should be considered as out- states while those with positive frequency should be considered as in-states, but at tree level analytical continuation from negative to positive frequency is trivial, so we do not distinguish them. The form-factors $< k_{J_1}, \ldots, k_{J_l} | \phi(x) | 0 >_{tree}$ in Eq.(3) are in the coordinate representation. The monomials in the harmonics $\mathcal{E}_J, J = 1, \ldots, L$ will produce the momentum conservation δ-functions after transformation to the momentum space.

It is very convenient to add to the above definition of the perturbiner the requirement of nilpotency of the symbols $a_J, J = 1, \ldots, L$, that is

$$a_J^2 = 0.^2 \tag{4}$$

In terms of the form-factors the nilpotency means that the form-factors with identical asymptotic states will not appear in the expansion of the perturbiner in powers of $a_J, J = 1, \ldots, L$ (see Eq.(3)). Clearly,

[1] In gauge theories the uniqueness is, of course, modulo gauge transformations

[2] This condition has nothing to do with the statistics of the particles considered. Say, for bosons the symbols still commute, while for fermions - anticommute.

it does not imply any loss of generality if the perturbiner is known for an arbitrary number L of the asymptotic one-particle states, since the form-factors with identical asymptotic states can be obtained from those with all asymptotic states different. It is clear that under the nilpotency assumption the perturbiner is, in fact, not the power series but just a polynomial in (nilpotent) harmonics $\mathcal{E}_J, J = 1, \ldots, L$ Eq.(2). Actually, for massless particles, the nilpotency assumption is necessary for the non-resonantness condition, because any multiple of a light-like momentum is again a light-like momentum, and the nilpotency makes these multiples irrelevant.

Everywhere below we assume the nilpotency condition Eq.(4).

We note that textbooks (see, e.g., [12], [13]) offer another definition of solution of field equations, generating tree amplitudes in the theory (they use the so-called Feynman asymptotic condition to define it).

Our definition above proved to be very convenient, in particular, it happened to be very conveniently compatible with the twistor description of solutions for the gauge self-duality equation ([8]), for the gravitational self-duality equations [9]) and for the gauge-gravitational self-duality equation ([10]). The traditional finite-action and reality conditions are substituted in the case of perturbiner by the condition of analyticity in the harmonics $\mathcal{E}_J, J = 1, \ldots, L$. Eqs.(1),(2). It is worth to explain that the perturbiner obeys the self-duality equations instead of the full equations, such as Yang-Mills equations or Einstein equations, when all polarizations entering Eq.(1) describe the same helicity state. Self-dual perturbiner generates only like-helicity form-factors. In this way the so-called Parke-Taylor amplitudes [4], [5] are very unusually described in terms of meromorphic functions on an auxiliary CP^1 space.

In this paper we describe perturbiner in $N = 3$ ($N = 4$) supersymmetric Yang-Mills theory. $N = 3$ supersymmetric Yang-Mills is equivalent to $N = 4$ super Yang-Mills but $N = 3$ superfield formalism is more naturally combined with twistors [1], that is why we follow $N = 3$ notation. $N = 4$ super Yang-Mills multiplet consists of the following particles:

$$\begin{aligned}1 \times 1 \,&(\text{positive helicity gluon})\\ 4 \times 1/2 \,&(\text{positive helicity gluinos})\\ 6 \times 0 \,&(\text{scalars})\\ 4 \times -1/2 \,&(\text{negative helicity gluinos})\\ 1 \times -1 \,&(\text{negative helicity gluon})\end{aligned} \tag{5}$$

This multiplet decomposes into two $N = 3$ multiplets as follows

$$\begin{aligned}1 &\times 1\\ 3 &\times 1/2\\ 3 &\times 0\\ 1 &\times -1/2\end{aligned} \tag{6}$$

$$\begin{aligned}1 &\times 1/2\\ 3 &\times 0\\ 3 &\times -1/2\\ 1 &\times -1\end{aligned} \tag{7}$$

It occurs that if one includes only states from one type of the $N = 3$ multiplets, say, the one from the table (6) (in arbitrary kinematics), that is all plane waves in Eq.(1) belong to the same type of the $N = 3$ multiplets (with arbitrary momenta $k_J, J = 1, \ldots, L$), the solution is obtained from the (non-supersymmetric) self-dual perturbiner [8] by substituting the harmonics $\mathcal{E}$ with their supersymmetric extensions $\mathcal{S}$. Such solution will be called chiral $N = 3$ perturbiner. If one includes both types of the $N = 3$ multiplets, the problem becomes much more complicated.[3] In this case we describe a twistor

[3] This is not surprising because all tree form-factors in non-supersymmetric Yang-Mills are contained among the $N = 4$ form-factors (when all external particles are gluons, the fermions and the charged scalars can appear only in loops). Actually, all $N = 3$ machinery can be viewed on as a convenient way to describe solutions of ordinary Yang-Mills equations, very much in the spirit of [1] (see, also, [2]).

reformulation of the problem, we show how it can be solved iteratively, but we have not been able to find a closed-form solution of the problem. Nevertheless, the problem is reduced to a neatly formulated algebraic geometry problem, Eqs.(26), (27),(28), and we feel that the complete solution might be somewhere nearby and we, perhaps, just do not know an appropriate mathematics to describe it.

My talk is organized as follows. In section 2 we remind $N=3$ super Yang-Mill equations and construct the plane wave solution of the linearized field equations. In section 3 we describe the chiral $N=3$ perturbiner. In section 4 the generic (non-chiral) perturbiner is considered.

2 N=3 supersymmetric perturbiner; preliminaries

$N=3$ and $N=4$ Yang-Mills theories are equivalent, but N=3 formalism is more naturally combined with the twistors [1] that is why we adopt $N=3$ notations. $N=3$ super-space is parametrized by commuting coordinates $x^{\alpha\dot\alpha}$ and by the anticommuting ones $\theta^{\alpha j}$, $\bar\theta^{\dot\alpha}_j$. $\alpha=1,2; \dot\alpha=\dot 1,\dot 2$ are the Lorentz indices, they can be lowered and raised with the antisymmetric ε-tensors, $j=1,2,3$ is an isotopic index. Introduce the following standard supersymmetric stuff $Q_{\alpha j}=\frac{\partial}{\theta^{\alpha j}}-\frac{1}{2}\bar\theta^{\dot\alpha}_j\partial_{\alpha\dot\alpha}$, $\bar Q^j_{\dot\alpha}=\frac{\partial}{\bar\theta^{\dot\alpha}_j}-\frac{1}{2}\theta^{\alpha j}\partial_{\alpha\dot\alpha}$, $D_{\alpha j}=\frac{\partial}{\theta^{\alpha j}}+\frac{1}{2}\bar\theta^{\dot\alpha}_j\partial_{\alpha\dot\alpha}$, $\bar D^j_{\dot\alpha}=\frac{\partial}{\bar\theta^{\dot\alpha}_j}+\frac{1}{2}\theta^{\alpha j}\partial_{\alpha\dot\alpha}$. Introduce also super-connections $\nabla_{\alpha j}=D_{\alpha j}+A_{\alpha j}$, $\bar\nabla^j_{\dot\alpha}=\bar D^j_{\dot\alpha}+\bar A^j_{\dot\alpha}$ where $A_{\alpha j}$ and $\bar A^j_{\dot\alpha}$ are superfields.

N=3 supersymmetric Yang-Mills equations can be represented as [1], [14]-[16]

$$\begin{aligned}(\nabla_{\alpha j}\nabla_{\beta l}+\nabla_{\beta l}\nabla_{\alpha j})\,|_{\text{symmetrized in }\alpha,\beta}&=0\\ \left(\bar\nabla^j_{\dot\alpha}\bar\nabla^l_{\dot\beta}+\bar\nabla^l_{\dot\beta}\bar\nabla^j_{\dot\alpha}\right)|_{\text{symmetrized in }\dot\alpha,\dot\beta}&=0\\ \nabla_{\alpha j}\bar\nabla^l_{\dot\beta}+\bar\nabla^l_{\dot\beta}\nabla_{\alpha j}&=\delta^l_j\nabla_{\alpha\dot\beta}\end{aligned}\tag{8}$$

(only traceless part of the last equation is, in fact, equation on $A_{\alpha j}$ and $\bar A^j_{\dot\alpha}$, the rest is the definition of connection $\nabla_{\alpha\dot\alpha}=\partial_{\alpha\dot\alpha}+A_{\alpha\dot\alpha}$).

As discussed in the Introduction, $N=4$ multiplet splits into two $N=3$ multiplets, see tables (5),(6),(7).

We first describe plane waves - solutions of the linearization of Eq.(8) - corresponding to the highest helicity states in each multiplet, that is the positive helicity gluon and $+\frac{1}{2}$ singlet gluino. These plane waves are

$$A_{\alpha j}=\frac{q_\alpha(\bar\theta_j,\bar{æ})t}{(æ,q)}e^{k_{\alpha\dot\alpha}y^{\alpha\dot\alpha}},\ \bar A^j_{\dot\alpha}=0\tag{9}$$

(positive helicity gluon)

$$A_{\alpha j}=0,\ \bar A^j_{\dot\alpha}=\frac{\bar q_{\dot\alpha}\frac{1}{2}\varepsilon^{jlm}(\bar\theta_l,\bar{æ})(\bar\theta_m,\bar{æ})t}{(\bar{æ},\bar q)}e^{k_{\alpha\dot\alpha}y^{\alpha\dot\alpha}}\tag{10}$$

($+\frac{1}{2}$ singlet gluino). In these equations $y^{\alpha\dot\alpha}=x^{\alpha\dot\alpha}+\frac{1}{2}\theta^{\alpha j}\bar\theta^{\dot\alpha}_j$ is a *chiral* coordinate, defined so that $\bar D^l_{\dot\beta}y^{\alpha\dot\alpha}=0$. The bracket (a,b), as before, stands for contraction of a and b with the ε-tensor, ε^{jlm} is the totally antisymmetric tensor in the isotopic space, $k^{\alpha\dot\alpha}=æ^\alpha\bar{æ}^{\dot\alpha}$ is a (light-like) four-momentum, t is a gauge Lie algebra generator, q_α and $\bar q_{\dot\alpha}$ are the reference spinors, they are defined modulo gauge equivalence $q_\alpha\sim q_\alpha+const\cdot æ_\alpha$, $\bar q_{\dot\alpha}\sim\bar q_{\dot\alpha}+const\cdot\bar{æ}_{\dot\alpha}$. One can check by a straightforward computation that the plane waves, Eqs.(9), (10) go through the linearized field equations Eqs.(8).

Acting with the $\bar Q$ charges on the states Eqs.(9) and Eqs.(10), one can obtain plane waves corresponding to all states in the tables Eqs.(6) and Eqs.(7). We shall, however, do a bit differently. Instead of plane waves of the type of $e^{k_{\alpha\dot\alpha}y^{\alpha\dot\alpha}}$, we shall use plane waves of the type of $e^{k_{\alpha\dot\alpha}y^{\alpha\dot\alpha}+(\theta^j,æ)\chi_j}$ which are proper states for the supercharges $Q_{\alpha j}$,

$$Q_{\beta l}e^{k_{\alpha\dot\alpha}y^{\alpha\dot\alpha}+(\theta^j,æ)\chi_j}=æ_\beta\chi_l e^{k_{\alpha\dot\alpha}y^{\alpha\dot\alpha}+(\theta^j,æ)\chi_j}\tag{11}$$

where $\chi_l, l=1,2,3$ are Grassmann variables; they are super-partners of momentum $k_{\alpha\dot\alpha}$, so they will be referred to as *momentino.*

Now the multiplets Eqs.(6) and Eqs.(7) can be organized as

$$A_{\alpha j} = \frac{q_\alpha((\bar{\theta}_j, \bar{æ}) + \chi_j)t}{(æ, q)} e^{k_{\alpha\dot{\alpha}} y^{\alpha\dot{\alpha}} + (\theta^j, æ)\chi_j}, \ \bar{A}^j_{\dot{\alpha}} = 0 \tag{12}$$

(positive helicity gluon)

$$A_{\alpha j} = 0, \ \bar{A}^j_{\dot{\alpha}} = \frac{\bar{q}\frac{1}{2}\varepsilon^{jlm}((\bar{\theta}_l, \bar{æ}) + \chi_l)((\bar{\theta}_m, \bar{æ}) + \chi_m)t}{(\bar{æ}, \bar{q})} e^{k_{\alpha\dot{\alpha}} y^{\alpha\dot{\alpha}} + (\theta^j, æ)\chi_j} \tag{13}$$

Various members of the multiplets Eqs.(6) and Eqs.(7) arise as coefficients in the Taylor expansion of Eqs.(12) and Eqs.(13) in powers of χ_j. For example, the plane wave solution corresponding to the negative helicity gluon state arises as coefficient at $\frac{1}{3!}\varepsilon^{jlm}\chi_j\chi_l\chi_m$ in the expansion of Eq.(13).

3 Chiral perturbiner

In this section we construct perturbiner which generates only form-factors with all asymptotic states belonging to the same type of the $N = 3$ super-multiplets, say, the one from the table (6) (no kinematical restrictions are assumed, that is, the set of momenta is arbitrary). According to the definition of the perturbiner (see Introduction), one first picks up a solution of the linearized field equations in the form of a superposition of plane waves,

$$A^{(1)}_{\alpha j} = \sum_{J=1}^{L} \frac{q^J_\alpha((\bar{\theta}_j, \bar{æ}^J) + \chi^J_j)t}{(æ^J, q^J)} \hat{\mathcal{S}}_J, \ \bar{A}^j_{\dot{\alpha}} = 0 \tag{14}$$

where $\hat{\mathcal{S}}_J = t_J \mathcal{S}$, $\mathcal{S} = a_J e^{k_{\alpha\dot{\alpha}} y^{\alpha\dot{\alpha}} + (\theta^j, æ^J)\chi^J_j}$, a^J is a commuting nilpotent (see Eq.(4)) symbol of annihilation/creation operator of the J-th particle, χ^J_j stands for the momentino of the J-th particle (see Eq(11)), other notations are as in Eqs.(9),(10).

Then one looks for a solution of the field equations (8), which is polynomial in the super-harmonics $\mathcal{S}_J, J = 1, \ldots, L$, and whose first order term is as in Eq.(14). To describe this solution, introduce again an auxiliary CP^1 space with homogeneous coordinates $\rho^\alpha, \alpha = 1, 2$. Introduce, also, D_j, ∇_j and A_j as $D_j = \rho^\alpha D_{\alpha j}$, $\nabla_j = \rho^\alpha \nabla_{\alpha j}$, $A_j = \rho^\alpha A_{\alpha j}$. Then, as in the self-dual case (section 2), the first equation of Eqs.(8) is represented as a zero-curvature condition by contracting its Lorence indices with ρ's. Hence, the first equation is (locally) solved as

$$A_j = g^{-1} D_j g \tag{15}$$

where (superfield) g on r.h.s. is a group valued rational homogeneous function of $\rho^\alpha, \alpha = 1, 2$ of degree 0, such that (superfield) A_j is a regular homogeneous function of $\rho^\alpha, \alpha = 1, 2$ of degree 1. The remaining equations of Eqs.(8) are solved provided

$$\bar{D}^j_{\dot{\alpha}} g = 0 \tag{16}$$

According to the rules of the game, g^{sptb} is sought of as a polynomial in the super-harmonics $\mathcal{S}_J, J = 1, \ldots, L$, first order term of the polynomial being defined by the one in $A_{\alpha j}$ Eq.(14) via Eq.(15). Amusingly, the resulting g^{sptb} is given by the same formulae as nonsupersymmetric self-dual g^{ptb} [8] with substitution of ordinary plane waves by the supersymmetric ones. Eq.(16) is satisfied simply because $\bar{D}^j\hat{\mathcal{S}} = 0$.

Finally, the generating functions for form-factors of the superfields $A_{\alpha j}$, $\bar{A}^j_{\dot{\alpha}}$ are obtained from Eqs.(15),

$$A^{sptb}_{\alpha j} = \sum_{J=1}^{L} A^J_{\alpha j}\hat{\mathcal{S}}_J + \sum J_1 J_2 A^{J_1 J_2}_{\alpha j}\hat{\mathcal{S}}_{J_1 J_2} + \ldots; \ \bar{A}^{sptb\, j}_{\dot{\alpha}} = 0 \tag{17}$$

where $A^{J_1 \ldots J_M}_{\alpha j} = -\frac{q_\alpha q^\beta\left([æ_\beta \chi_j]^{J_1 \ldots J_M} + \bar{\theta}^{\dot{\alpha}} k^{J_1 \ldots J_M}_{\beta\dot{\alpha}}\right)}{(æ_{J_1}, q)(æ_{J_M}, q)} \cdot \frac{1}{(æ^{J_1}, æ^{J_2})(æ^{J_2}, æ^{J_3})\ldots(æ^{J_{M-1}}, æ^{J_M})}$ and $[æ_\beta \chi_j]^{J_1 \ldots J_M} = æ^{J_1}_\beta \chi^{J_1}_j + \ldots + æ^{J_M}_\beta \chi^{J_M}_j$.

4 Generic (nonchiral) perturbiner

Nonchiral $N = 3$ supersymmetric perturbiner is a generating function for all tree form-factors in the $N = 3$ ($N = 4$) supersymmetric Yang-Mills theory. According to the rules of the game, the solution of the linearized field equations must include now both types of harmonics, Eqs.(12),(13),

$$A^{(1)}_{\alpha j} = \sum_{J=1}^{L} \frac{q^J_\alpha((\bar\theta_j, \bar{æ}^J) + \chi^J_j)t}{(æ^J, q^J)}\hat{\mathcal{S}}_J$$

$$\bar{A}^{(1)j}_{\dot\alpha} = \sum_{J=1}^{L} b_J \frac{\bar{q}^J_{\dot\alpha}\frac{1}{2}\varepsilon^{jlm}((\bar\theta_l, \bar{æ}^J) + \chi^J_l)((\bar\theta_m, \bar{æ}^J) + \chi^J_m)t}{(\bar{æ}^J, \bar{q}^J)}\hat{\mathcal{S}}_J \tag{18}$$

where b_J is an anticommuting nilpotent symbol, all other notations are the same as in Eq.(14).

The nonchiral perturbiner is a solution of the field equations Eqs.(8), polynomial in the harmonics $\mathcal{S}_J, J = 1, \ldots, L$, the first order term of the polynomial being just $A^{(1)}$ as in Eq.(14). We again use a twistor construction [1] to describe solutions of Eqs.(8). To this end, introduce two couples of complex numbers, $\rho^\alpha, \alpha = 1, 2$, $\bar\rho^{\dot\alpha}, \dot\alpha = \bar{1}, \bar{2}$ which can be viewed of as homogeneous coordinates on $CP^1 \times CP^1$ space. Contracting all Lorence indices in Eqs.(8) with these of ρ's, $\bar\rho$'s, one again obtains a sort of zero-curvature condition, which is locally solved as

$$A_j = g^{-1} D_j g,\ \bar{A}^j = g^{-1}\bar{D}^j g \tag{19}$$

where $A_j = \rho^\alpha A_{\alpha j}$, $D_j = \rho^\alpha D_{\alpha j}$, $\bar{A}^j = \bar\rho^{\dot\alpha}\bar{A}^j_{\dot\alpha}$, $\bar{D}^j = \bar\rho^{\dot\alpha}\bar{D}^j_{\dot\alpha}$, and ρ, $\bar\rho$ dependence of g must be such that A_j, $\bar{A}^j$ are just linear in ρ, $\bar\rho$. More definitely, (superfield) g is a meromorphic homogeneous function of ρ, $\bar\rho$ of degree $(0, 0)$ such that (superfield) A_j is a holomorphic homogeneous function of ρ, $\bar\rho$ of degree $(1, 0)$ and (superfield) $\bar{A}^j$ is a holomorphic homogeneous function of ρ, $\bar\rho$ of degree $(0, 1)$.

In the case of perturbiner, g^{ncsptb} must be polynomial in the super-harmonics $\mathcal{S}_J, J = 1, \ldots, L$. First order terms of this polynomial,

$$g^{(1)}_{ncsptb}(\rho, \bar\rho, \{\mathcal{S}\}) = \sum_{J=1}^{L} g^{ncsptb}_J(\rho, \bar\rho)\hat{\mathcal{S}}_J \tag{20}$$

are fixed by those in A's Eqs.(18) via Eqs.(19) . Expanding Eq.(19) up to the first order in the harmonics $\mathcal{S}_J$ and using Eq.(18),Eq.(20) one obtains $g^{ncsptb}_J(\rho, \bar\rho) = \frac{(\rho, q^J)}{(\rho, æ^J)(æ^J, q^J)} + b^J \frac{(\bar\rho, \bar{q}^J)}{(\bar\rho, \bar{æ}^J)(\bar{æ}^J, \bar{q}^J)} \cdot \frac{1}{3!}\varepsilon^{jlm}((\bar\theta_l, \bar{æ}^J) + \chi^J_j)((\bar\theta_l, \bar{æ}^J) + \chi^J_l)((\bar\theta_m, \bar{æ}^J) + \chi^J_m)$ Thus, according to Eqs.(20), the first order terms in g^{ncsptb} have simple poles at surfaces

$$\rho_\alpha = æ^J_\alpha,\ J = 1, \ldots, L\ and\ \bar\rho_{\dot\alpha} = \bar{æ}^J_{\dot\alpha},\ J = 1, \ldots, L \tag{21}$$

in the $CP^1 \times CP^1$ space parametrized by ρ, $\bar\rho$. An analysis shows that for A, $\bar{A}$ from Eqs.(19) to be regular, the higher order terms in the polynomial g_{ncsptb} may have only simple poles at the surfaces Eqs.(21) and also at the surfaces $C_{\mathcal{M}}$ in $CP^1 \times CP^1$

$$C_{\mathcal{M}} : \{\rho^\alpha \bar\rho^{\dot\alpha} k^{\mathcal{M}}_{\alpha\dot\alpha} = 0\} \tag{22}$$

where $k^{\mathcal{M}}_{\alpha\dot\alpha} = \sum_{J\in\mathcal{M}} k^J_{\alpha\dot\alpha}$, and $\mathcal{M}$ is a subset of the set $J = 1, \ldots, L$. That is, any linear combination of momenta of the asymptotic states included in the perturbiner defines a surface in $CP^1 \times CP^1$ at which g_{ncsptb} has a simple pole. Notice that due to the non-resonantness condition (see Introduction) the surfaces $C_{\mathcal{M}}$ Eq.(22) with $\mathcal{M}$ including more than one element never reduce to the ones of the type of Eqs.(21). Furthermore, the regularity of A, $\bar{A}$ from Eqs.(19) dictates a condition on residues of g_{ncsptb} at the surfaces $C_{\mathcal{M}}$, namely

$$\Big(D_j(res|_{C_{\mathcal{M}}} g_{ncsptb} \cdot g^{-1}_{ncsptb})\Big)|_{C_{\mathcal{M}}} = 0,\ \Big(\bar{D}^j res|_{C_{\mathcal{M}}} g_{ncsptb} \cdot g^{-1}_{ncsptb})\Big)|_{C_{\mathcal{M}}} = 0 \tag{23}$$

where notation $|_{C_{\mathcal{M}}}$ in Eqs.(23) indicates restriction at the surface $C_{\mathcal{M}}$ Eq.(22). It is convenient to use the expansion of g_{ncsptb} in the color-ordered monomials,

$$g_{ncsptb}(\rho, \bar\rho, \{\mathcal{S}\}) = 1 + \sum_J g^{ncsptb}_J(\rho\bar\rho)\hat{\mathcal{S}}_J + \sum_{J_1, J_2} g^{ncsptb}_{J_1, J_2}(\rho, \bar\rho)\hat{\mathcal{S}}_{J_1}\hat{\mathcal{S}}_{J2} + \ldots \tag{24}$$

The singularity structure of g_{ncsptb} Eqs.(22),(23) dictates the following form of coefficients $g^{ncsptb}_{J_1,J_2,\ldots,J_L}(\rho,\bar\rho)$,

$$g^{ncsptb}_{J_1,J_2,\ldots,J_M}(\rho,\bar\rho)=\frac{P^{J_1J_2\ldots J_M}(\rho,\bar\rho,\bar\theta)}{(\rho^\alpha\bar\rho^{\dot\alpha}k^{J_1}_{\alpha\dot\alpha})(\rho^\alpha\bar\rho^{\dot\alpha}k^{J_1J_2}_{\alpha\dot\alpha})\ldots(\rho^\alpha\bar\rho^{\dot\alpha}k^{J_1J_2\ldots J_M}_{\alpha\dot\alpha})} \tag{25}$$

where, we remind, $k^{J_1J_2\ldots J_M}_{\alpha\dot\alpha}=k^{J_1}_{\alpha\dot\alpha}+k^{J_2}_{\alpha\dot\alpha}+\ldots+k^{J_M}_{\alpha\dot\alpha}$, and $P^{J_1J_2\ldots J_M}(\rho,\bar\rho,\bar\theta)$ is a polynomial in ρ, $\bar\rho$ of degree (M,M) (so, that $g^{ncsptb}_{J_1,J_2,\ldots,J_M}(\rho,\bar\rho)$ is a rational function of ρ, $\bar\rho$ of degree $(0,0)$). We have explicitly indicated only ρ, $\bar\rho$ and $\bar\theta$ dependence of these polynomials. They, of course, depend on quantum numbers of J_1-th, J_2-th,...,J_M-th particles, such as momenta, momentinos etc.

Now, the problem of constructing g_{ncsptb}, and thus the problem of computing all tree form-factors in (supersymmetric) Yang-Mills theory will be formulated as a problem of constructing a set of polynomials $P^{J_1J_2\ldots J_M}(\rho,\bar\rho,\bar\theta)$, $M=1,\ldots,L$. From Eqs.(23),(24) and (25), crucially using the nilpotency, $\mathcal{S}^2=0$, and linear independence of various color ordered products of $\mathcal{S}$'s, one obtains

$$\begin{aligned}\left(\rho^\alpha([æ_\alpha\chi_j]^{J_1\ldots J_M}+\bar\theta^{\dot\alpha}_j k^{J_1J_2\ldots J_M}_{\alpha\dot\alpha})P^{J_1J_2\ldots J_M}\right)|_{C_{J_1J_2\ldots J_M}}&=0\\ \left(\bar\rho^{\dot\alpha}\frac{\partial}{\bar\theta^{\dot\alpha}_j}P^{J_1J_2\ldots J_M}\right)|_{C_{J_1J_2\ldots J_M}}&=0\end{aligned} \tag{26}$$

$$P^{J_1J_2\ldots J_M}|_{C_{J_1J_2\ldots J_I}}=P^{J_1J_2\ldots J_I}|_{C_{J_1J_2\ldots J_I}}P^{J_{I+1}\ldots J_M}|_{C_{J_1J_2\ldots J_I}},\ I<M, \tag{27}$$

and from Eqs.(20) one sees that

$$\begin{aligned}P^J=\frac{(\rho,q^J)(\bar\rho,\bar q^J)}{(æ^J,q^J)}+b^J\frac{(\bar\rho,\bar q^J)(\rho,æ^J)}{(\bar æ^J,\bar q^J)}\cdot\\ \frac{1}{3!}\varepsilon^{jlm}((\bar\theta_j,\bar æ^J)+\chi^J_j)((\bar\theta_l,\bar æ^J)+\chi^J_l)((\bar\theta_m,\bar æ^J)+\chi^J_m)\end{aligned} \tag{28}$$

$k^{J_1J_2\ldots J_L}_{\alpha\dot\alpha}=k^{J_1}_{\alpha\dot\alpha}+k^{J_2}_{\alpha\dot\alpha}+\ldots+k^{J_L}_{\alpha\dot\alpha}$, ($k^J_{\alpha\dot\alpha}=æ^J_\alpha\bar æ^J_{\dot\alpha}$ is momentum of the J-th particle). Notation $P|_C$ means that a polynomial P is restricted of a surface C. $C_{J_1J_2\ldots J_M}$ are surfaces in $CP^1\times CP^1$ defined by Eqs.(22). Due to the non-resonantness condition (that a sum of the type of $k^{J_1}_{\alpha\dot\alpha}+k^{J_2}_{\alpha\dot\alpha}+\ldots+k^{J_L}_{\alpha\dot\alpha}$ is never light-like when there is more than one term) only surfaces C_J are reducible, as in Eqs.(21). Notice that the surfaces $C_{J_1J_2\ldots J_I}$ and $C_{J_{I+1}J_2\ldots J_L}$, $I<L$ intersect at two points in $CP^1\times CP^1$, defined by the system

$$\rho^\alpha\bar\rho^{\dot\alpha}k^{J_1J_2\ldots J_I}_{\alpha\dot\alpha}=0,\ \rho^\alpha\bar\rho^{\dot\alpha}k^{J_{I+1}J_2\ldots J_L}_{\alpha\dot\alpha}=0 \tag{29}$$

and at every one of these intersection points three surfaces meet - $C_{J_1J_2\ldots J_I}$, $C_{J_{I+1}J_2\ldots J_L}$ and $C_{J_1J_2\ldots J_L}$.

We shall now explain how this problem, Eqs.(26), (27),(28), can be solved iteratively. Namely, we explain how to find polynomial $P^{J_1J_2\ldots J_L}$, provided all polynomials $P^{J_1J_2\ldots J_I}$, $I<L$, are known (in this sense Eq.(28) is a first step of the iteration).

Consider, first, the polynomials restricted to "their own" surfaces, that is, introduce $\tilde P^{J_1J_2\ldots J_L}=P^{J_1J_2\ldots J_L}|_{C_{J_1J_2\ldots J_L}}$. $\tilde P^{J_1J_2\ldots J_L}$ is a homogeneous polynomial on $C_{J_1J_2\ldots J_L}$ of degree $2L$, and hence it has $2L+1$ degrees of freedom.

Equations (27) on the restricted polynomials reduce to

$$\begin{aligned}\tilde P^{J_1J_2\ldots J_L}|_{Z_i(C_{J_1J_2\ldots J_I}\cap C_{J_{I+1}\ldots J_L})}=P^{J_1J_2\ldots J_I}|_{Z_i(C_{J_1J_2\ldots J_I}\cap C_{J_{I+1}\ldots J_L})}\\ P^{J_{I+1}\ldots J_L}|_{Z_i(C_{J_1J_2\ldots J_I}\cap C_{J_{I+1}\ldots J_L})},\ I<L,\end{aligned} \tag{30}$$

where $Z_i(C_{J_1J_2\ldots J_I}\cap C_{J_{I+1}\ldots J_L}), i=1,2$ stand for the intersection points of the surfaces $C_{J_1J_2\ldots J_I}$ and $C_{J_{I+1}\ldots J_L}$, Eq.(29). This defines polynomial $\tilde P^{J_1J_2\ldots J_L}$ at these $2(L-1)$ intersection points.

Equations (26) tell that the polynomial $\tilde P^{J_1J_2\ldots J_L}$ must be of the form

$$\begin{aligned}\tilde P^{J_1J_2\ldots J_L}(\rho,\bar\rho,\bar\theta)=\frac{1}{3!}\varepsilon^{jlm}\left(\rho^\alpha([æ_\alpha\chi_j]^{J_1\ldots J_M}+\bar\theta^{\dot\alpha}_j k^{J_1J_2\ldots J_M}_{\alpha\dot\alpha})\right)\\ \left(\rho^\alpha([æ_\alpha\chi_l]^{J_1\ldots J_M}+\bar\theta^{\dot\alpha}_l k^{J_1J_2\ldots J_M}_{\alpha\dot\alpha})\right)\left(\rho^\alpha([æ_\alpha\chi_m]^{J_1\ldots J_M}+\bar\theta^{\dot\alpha}_m k^{J_1J_2\ldots J_M}_{\alpha\dot\alpha})\right)\\ R^{J_1J_2\ldots J_L}(\rho,\bar\rho)|_{C_{J_1J_2\ldots J_L}}\end{aligned} \tag{31}$$

where $R^{J_1 J_2 \dots J_L}(\rho,\bar\rho)|_{C_{J_1 J_2 \dots J_L}}$ is a polynomial on the surface $C_{J_1 J_2 \dots J_L}$ of degree $2L-3$. Thus, the number of degrees of freedom in the polynomial $R^{J_1 J_2 \dots J_I}(\rho,\bar\rho)|_{C_{J_1 J_2 \dots J_L}}$, and so in the polynomial $\tilde P^{J_1 J_2 \dots J_I}$, is $2L-2$, that is precisely the number of points at which the polynomial is defined according to Eqs.(30)! (Eqs.(31) and (30) are compatible due to nilpotency of $\theta^{\alpha j}$, $\bar\theta^{\dot\alpha}_j$ and χ^J_j).

Once restriction of the polynomial $P^{J_1 J_2 \dots J_L}$ to the surface $C_{J_1 J_2 \dots J_L}$ is found and restrictions of the polynomial $P^{J_1 J_2 \dots J_L}$ on the surfaces $C_{J_1 J_2 \dots J_M}$, $M<L$ are known due to equations (27), the polynomial $P^{J_1 J_2 \dots J_L}$ is fixed modulo a polynomial which is zero at surfaces $C_{J_1 J_2 \dots J_M}$, $M=1,\dots,L$, that is precisely modulo a polynomial in the denominator of $g^{ncsptb}_{J_1,J_2,\dots,J_L}$, Eq.(25), that is, modulo the gauge freedom.

The described iterative procedure can, in principle, be used as an alternative to the usual perturbation theory, and it might even be the more economical alternative, but we shall not try here to prove its efficiency. Instead, we would like to express our hope that Eqs.(26), (27),(28) can be, in some sense, solved completely. By the way, Eqs.(27),(28) allow such a complete solution up to a freedom which is to be fixed by Eqs.(26) (or Eqs.(31), namely

$$P^{J_1 J_2 \dots J_L} = \det \begin{vmatrix} P^{J_1} & Q^{J_1 J_2} & Q^{J_1 J_2 J_3} & \dots & Q^{J_1 \dots J_L} \\ \rho^\alpha \bar\rho^{\dot\alpha} k^{J_1}_{\alpha\dot\alpha} & P^{J_2} & Q^{J_2 J_3} & \dots & Q^{J_2 \dots J_L} \\ 0 & \rho^\alpha \bar\rho^{\dot\alpha} k^{J_1 J_2}_{\alpha\dot\alpha} & P^{J_3} & \dots & Q^{J_3 \dots J_L} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & P^{J_L} \end{vmatrix}$$

Each entry of the $L\times L$ matrix in the above equation is a homogeneous polynomial on the $CP^1\times CP^1$ space of degree $(1,1)$. Q's in the upper triangular part of the matrix represent the freedom which is to be fixed by Eqs.(26) (or Eqs.(31)). Unfortunately, we have not been able to implement these equations to fix this freedom.

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