Parameter Symmetry of Boson Models

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Abstract

A symmetry of the parameter space of Interacting Boson Models IBM-1 and IBM-2 is studied. The symmetry is associated with linear canonical transformations of boson operators, or, equivalently, with the existence of different realizations of the symmetry algebras of the models. The relevance of the parameter symmetry to physical observables is discussed.

1 Introduction

Algebraic models with Hamiltonian or Lagrangian which structure is dictated by the underlying symmetry of the system, are popular in various fields of physics, in particular, in high energy physics, nuclear and molecular physics [1]. Recently it was found [2, 3, 4] that there is a non-trivial symmetry in the parameter space of such models hereafter referred to as a parameter symmetry (PS). The parameter symmetry gives rise to various hidden dynamical symmetries (DS) [5] of the corresponding model.

In this paper we discuss the PS concept using as an example the simplest version of the Interacting Boson Model IBM-1 [6, 1] and extend it on the case of a more complicated version of the same model IBM-2 [6, 1]. Both IBM-1 and IBM-2 are actively used in modern nuclear applications. We show that PS is associated with the ambiguity in definition of boson operators and with a possibility to arrange different realizations of the symmetry algebra. The manifestation of PS in physical observables is discussed. In the case of a model with a complicated symmetry algebra like IBM-2, PS is shown to be useful for constructing the general Hamiltonian of the model.

The parameter symmetry is a common property of boson models. For example, we have shown [7] that the vibron model used in molecular and high-energy applications (see e.g. [1]) possesses some PS; we suppose that PS can be found in other algebraic models including fermion and boson-fermion ones.

2 Definitions

Let G be a symmetry group of the model. We suppose also that all eigenstates can be labelled by quantum numbers associated with some subgroup $\mathrm{G}_{\mathrm{inv}} \subset \mathrm{G}$ [for example, if the rotational symmetry is adopted, all eigenstates can be labelled by angular momentum L and its projection m, and $\mathrm{G}_{\mathrm{inv}}$ is SO(3) or SU(2)]. Therefore we are interested in reduction chains of the group G (or of the corresponding algebra G) of the type

$$\mathrm{G} \supset \begin{array}{l} \mathrm{G}_1 \supset \mathrm{G}_2 \supset \mathrm{G}_3 \supset \ldots \supset \mathrm{G}_{inv} \\ \widetilde{\mathrm{G}}_1 \supset \widetilde{\mathrm{G}}_2 \supset \widetilde{\mathrm{G}}_3 \supset \ldots \supset \mathrm{G}_{inv} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \widetilde{\widetilde{\mathrm{G}}}_1 \supset \widetilde{\widetilde{\mathrm{G}}}_2 \supset \widetilde{\widetilde{\mathrm{G}}}_3 \supset \ldots \supset \mathrm{G}_{inv} \end{array} . \tag{1}$$

The general Hamiltonian of the model is a superposition of Casimir operators of all subalgebras of the algebra G entering the reduction chains (1):

$$H(\{k\}) = H_0 + \sum_i \left[k_i C(\mathrm{G}_i) + \tilde{k}_i C\left(\widetilde{\mathrm{G}}_i\right) + \widetilde{\widetilde{k}}_i C\left(\widetilde{\widetilde{\mathrm{G}}}_i\right) + \ldots \right]. \tag{2}$$

The Hamiltonian $H(\{k\})$ is a function of the set of its parameters $\{k\} \equiv \{k_1, k_2, \ldots, \tilde{k}_1, \tilde{k}_2, \ldots\}$. The situation when for some more or less general set of parameters $\{k\}$ there exists another set $\{k'\} \equiv$

Table 1: Generators of U(6), U(5), SU(3), SO(6), SO(5) and SO(3) algebras.

Algebra	Generators
U(6)	$[d^+\times s]^{(2)}_\mu,\ [s^+\times\tilde{d}]^{(2)}_\mu,\ S=[s^+\times s]^{(0)}_0,\ T^\lambda_\mu=[d^+\times\tilde{d}]^{(\lambda)}_\mu,\ \lambda=0,1,2,3,4$
U(5)	$T^\lambda_\mu=[d^+\times\tilde{d}]^{(\lambda)}_\mu,\quad \lambda=0,1,2,3,4$
SU(3)	$Q_\mu=[d^+\times s+s^+\times\tilde{d}]^{(2)}_\mu-\frac{\sqrt{7}}{2}[d^+\times\tilde{d}]^{(2)}_\mu,\ T^1_\mu=[d^+\times\tilde{d}]^{(1)}_\mu$
SO(6)	$Q^0_\mu=[d^+\times s+s^+\times\tilde{d}]^{(2)}_\mu,\ T^\lambda_\mu=[d^+\times\tilde{d}]^{(\lambda)}_\mu,\ \lambda=1,3$
SO(5)	$T^\lambda_\mu=[d^+\times\tilde{d}]^{(\lambda)}_\mu,\quad \lambda=1,3$
SO(3)	$T^1_\mu=[d^+\times\tilde{d}]^{(1)}_\mu$

$\{k'_1,k'_2,\dots,\tilde{k}'_1,\tilde{k}'_2,\dots\}$ such that the Hamiltonians $H(\{k\})$ and $H(\{k'\})$ have identical spectra of eigenvalues, is referred to as parameter symmetry.

The dynamical symmetry limit corresponds to the case when the Hamiltonian $H(\{k\})$ involves Casimir operators belonging to one of the reduction chains (1) only, and hence the eigenvalues and eigenfunctions can be found analytically. Obviously, the DS limit is associated with a specific choice of the set of Hamiltonian parameters $\{k\}$ when some of them take zero values, e.g. $\{k\}\equiv\{k_1,k_2,\dots,\ \tilde{k}_1{=}0,\tilde{k}_2{=}0,\dots,\ \tilde{\tilde{k}}_1{=}0,\tilde{\tilde{k}}_2{=}0,\dots\}$. Due to PS the DS limit Hamiltonian $H(\{k\})$ may appear to be isospectral to some Hamiltonian $H(\{k'\})$ that does not correspond to any of DS limits. This situation is referred to as hidden (dynamical) symmetry [5].

3 Parameter Symmetry of IBM-1

Within IBM-1 nuclear states are labelled by a fixed total number N of bosons of two types, s and d, with quantum numbers $l^\pi=0^+$ and $l^\pi=2^+$, respectively [6]. 36 bilinear combinations of boson operators s^+s, $d^+_\mu d_\nu$, $d^+_\mu s$, s^+d_μ generate the U(6) algebra. The general IBM-1 Hamiltonian is expressed as [6]

$$H(\{k_i\}) = H_0+k_1C_1[\mathrm{U}(5)]+k_2C_2[\mathrm{U}(5)]+k_3C_2[\mathrm{SO}(5)]+k_4C_2[\mathrm{SO}(3)]+k_5C_2[\mathrm{SO}(6)]+k_6C_2[\mathrm{SU}(3)]\,, \tag{3}$$

where C_1 and C_2 stand for the first and the second rank Casimir invariants of the algebras entering the following reduction chains of the U(6) algebra:

$$\begin{array}{lll} & \mathrm{U}(5)\supset\mathrm{SO}(5)\supset\mathrm{SO}(3) & \mathrm{I}\\ \nearrow & & \\ \mathrm{U}(6)\supset\mathrm{SU}(3)\supset\mathrm{SO}(3) & & \mathrm{II}\\ \searrow & & \\ & \mathrm{SO}(6)\supset\mathrm{SO}(5)\supset\mathrm{SO}(3) & \mathrm{III}\end{array} \tag{4}$$

We define the Casimir operators as in the book [1]:

$$\begin{array}{lll} C_1[\mathrm{U}(5)]=n_d\,, & C_2[\mathrm{U}(5)]=n_d(n_d{+}4), & C_2[\mathrm{SU}(3)]=2(Q\cdot Q)+\frac{15}{2}(T^1\cdot T^1),\\ C_2[\mathrm{SO}(3)]=10(T^1\cdot T^1), & C_2[\mathrm{SO}(5)]=2(T^3{\cdot}T^3)+2(T^1\cdot T^1), & C_2[\mathrm{SO}(6)]=N(N{+}4)-4\,(P^+\cdot P)\,,\end{array} \tag{5}$$

where generators of the U(6) algebra and its subalgebras are given in Table 1,

$$n_d=(d^+\cdot\tilde{d}),\quad P=\tfrac{1}{2}\Big((\tilde{d}\cdot\tilde{d})-(s\cdot s)\Big)\,, \tag{6}$$

$\tilde{d}_\mu=(-1)^\mu d_{-\mu}$, $(t\cdot u)$ and $[t\times u]^{(\lambda)}_\mu$ are scalar and tensor products, respectively, of spherical tensors t and u.

The DS limits of the IBM-1 Hamiltonian discribe typical nuclear spectra [6]: vibrational in the U(5) DS limit ($k_5{=}k_6{=}0$), rotational in the SU(3) DS limit ($k_1{=}k_2{=}k_3{=}k_5{=}0$) and γ-unstable in the SO(6) DS

limit (k_1=k_2=k_6=0). A transitional nuclear Hamiltonian that does not possess any DS, is conventionally believed to generate a spectrum different from those corresponding to any of the DS limits.

As we have shown in Refs. [2, 3, 4], the IBM-1 Hamiltonian possesses the following PS:

Hamiltonians $H(\{k_i\})$ and $H(\{k'_i\})$ defined by Eq. (3) *are isospectral if the corresponding parameter sets $\{k_i\}$ and $\{k'_i\}$ are related as*

$$H'_0 = H_0,\ k'_1 = k_1 + 2k_6,\ k'_2 = k_2 + 2k_6,\ k'_3 = k_3 - 6k_6,\ k'_4 = k_4 + 2k_6,\ k'_5 = k_5 + 4k_6,\ k'_6 = -k_6 \quad (7)$$

in the case $k_6 \neq 0$, or as

$$H'_0 = H_0 + 10k_5N,\ k'_1 = k_1 + 4k_5(N{+}2),\ k'_2 = k_2 - 4k_5,\ k'_3 = k_3 + 2k_5,\ k'_4 = k_4,\ k'_5 = -k_5,\ k'_6 = 0 \quad (8)$$

in the case $k_6 = 0$.

This statement was proved (see Refs. [3, 2, 4] for details) by constructing a unitary transformation U such that $H(\{k'_i\}) = UH(\{k_i\})U^{-1}$.

Thus, for any set of the IBM-1 parameters there is another set which generates the identical spectrum. The only exception is the U(5) DS limit when the two sets of the parameters coincide as is seen from (8).

The PS relations (7) and (8) give rise to IBM-1 hidden DS. As follows from Eqs. (7), the rotational spectrum of the SU(3) DS limit (k_1=k_2=k_3=k_5=0) appears to be equivalent to the spectrum of the transitional Hamiltonian with the set of parameters $\{k'_i\} \equiv \{k'_1{=}2k_6,\ k'_2{=}2k_6,\ k'_3{=}{-}6k_6,\ k'_4{=}k_4{+}2k_6,\ k'_5{=}2k_6,\ k'_6{=}{-}k_6\}$ that does not correspond to any DS. Similarly, it follows from (8), that the γ-unstable spectrum of the SO(6) DS limit (k_1=k_2=k_6=0) can be obtained with the set of parameters $\{k'_i\} \equiv \{k'_6{=}0,\ k'_1{=}8(N{+}2)k_5,\ k'_2{=}{-}8k_5,\ k'_3{=}k_3{+}4k_5,\ k'_4{=}k_4,\ k'_5{=}{-}k_5\}$ corresponding to the U(5)–SO(6) transitional nuclear spectrum.

To reveal the origin of PS, we note that there is an ambiguity in definition of boson operators within IBM [4, 6, 1, 8, 9]. One can apply to the boson operators gauge transformations $R_s(\varphi_s)$ and $R_d(\varphi_d)$ defined as [8, 6]

$$\begin{aligned} R_s(\varphi_s)\, s^+ &= \exp(\mathrm{i}\varphi_s/2)\, s^+, & R_s(\varphi_s)\, s &= \exp(-\mathrm{i}\varphi_s/2)\, s\,, \\ R_d(\varphi_d)\, d^+_\mu &= \exp(\mathrm{i}\varphi_d/2)\, d^+_\mu\,, & R_d(\varphi_d)\, \tilde{d}_\mu &= \exp(-\mathrm{i}\varphi_d/2)\, \tilde{d}_\mu\,. \end{aligned} \quad (9)$$

Note that these transformations are the canonical ones, i.e. they do not violate the boson commutation relations. However, the structure of the IBM-1 Hamiltonian implies severe restrictions on the use of the transformations (9). For example, in the case of the general IBM-1 Hamiltonian, one can use the gauge transformation $R(\varphi) \equiv R_s(\varphi_s) \times R_d(\varphi_d)$ with $\varphi \equiv (\varphi_s - \varphi_d)/2 = 0,\ \pi$ only and arbitrary $\tilde{\varphi} \equiv (\varphi_s + \varphi_d)/2$ [8, 9]. In the case of the transitional SO(6)–U(5) IBM-1 Hamiltonian with $k_6 = 0$, the gauge transformation $R(\varphi)$ with $\varphi = 0,\ \dfrac{\pi}{2},\ \pi,\ \dfrac{3\pi}{2}$ and arbitrary $\tilde{\varphi}$ can be used. One can also apply to the boson operators the particle-hole conjugation $\tilde{R}$ [10, 1, 9] defined as

$$\tilde{R}\, s^+ = s\,, \qquad \tilde{R}\, s = -s^+, \qquad \tilde{R}\, d^+_\mu = \tilde{d}_\mu\,, \qquad \tilde{R}\, \tilde{d}_\mu = -d^+_\mu\,, \quad (10)$$

and operators $\tilde{R}(\varphi) \equiv \tilde{R} \times R(\varphi)$ that are consistent with the Hamiltonian structure provided that $\varphi = 0,\ \dfrac{\pi}{2},\ \pi,\ \dfrac{3\pi}{2}$ in the case $k_6 = 0$ or $\varphi = 0,\ \pi$ in the case $k_6 \neq 0$. The operators $R(\varphi)$ and $\tilde{R}(\varphi)$ comprise a point group studied elsewhere [9].

We use the following notations for operators subjected to the transformations $R(\varphi)$ and $\tilde{R}(\varphi)$: ${}^{\alpha}O \equiv R(\varphi)\, O$ and ${}^{-\alpha}O \equiv \tilde{R}(\varphi)\, O$, $\alpha = \varphi/\pi$.

Hamiltonians ${}^{\alpha}H(\{k_i\}) \equiv R(\varphi)\, H(\{k_i\})$ and the initial Hamiltonian $H(\{k_i\})$ are, of course, isospectral [Hamiltonians ${}^{-\alpha}H(\{k_i\}) \equiv \tilde{R}(\varphi)\, H(\{k_i\})$ may be not isospectral to $H(\{k_i\})$; however, one can always find a simple and not very restrictive constraint on the parameters k_i that will guarantee the isospectrality of ${}^{-\alpha}H(\{k_i\})$ and $H(\{k_i\})$]. Thus we can use transformations (9) [and in some cases (10)] to study PS and hidden symmetries of IBM.

For example, the Hamiltonian ${}^{1}H(\{k_i\}) \equiv R(\pi)\, H(\{k_i\})$ is isospectral but not equivalent to the initial Hamiltonian $H(\{k_i\})$. Using Table 1 and expressions (3), (5), (6) and (9), one can obtain [4] that ${}^{1}H(\{k_i\}) = H(\{k'_i\})$ where the set of parameters $\{k'_i\}$ is defined by the PS relation (7). Thus, the transformation $R(\pi)$ is equivalent to the PS transformation (7).

Note that, as is seen from Eqs. (5)–(6) and Table 1, only the Casimir operator $C_2[\mathrm{SU}(3)]$ and the SU(3) generator Q_μ are changed under the transformation $R(\pi)$:

$$R(\pi)\, Q_\mu = {}^1Q_\mu = -[d^+ \times s \,+\, s^+ \times \tilde{d}]^{(2)}_\mu - \frac{\sqrt{7}}{2}[d^+ \times \tilde{d}]^{(2)}_\mu . \tag{11}$$

The quadrupole operators Q_μ and ${}^1Q_\mu$ correspond to different embeddings of the SU(3) subalgebra in the U(6) algebra [see also [10] for other realizations of SU(3)]. Using PS transformation (7) it is easy to express the Casimir operator $C_2\left[\mathrm{SU}^1(3)\right]$ of the $\mathrm{SU}^1(3)$ algebra associated with the quadrupole operator (11) through $C_2\left[\mathrm{SU}(3)\right]$ and Casimir operators of other algebras [2, 4]:

$$C_2\left[\mathrm{SU}^1(3)\right] = 2C_1[\mathrm{U}(5)] + 2C_2[\mathrm{U}(5)] - 6C_2[\mathrm{SO}(5)] + \, 2C_2[\mathrm{SO}(3)] + 4C_2[\mathrm{SO}(6)] - C_2[\mathrm{SU}(3)] \, . \tag{12}$$

In the case $k_6 = 0$, ${}^1H(\{k_i\}) \equiv H(\{k_i\})$ and the transformation $R(\pi)$ does not generate PS. However, in this case we can apply the transformation $R(\pi/2)$ to the Hamiltonian. Using Table 1 and expressions (3), (5), (6) and (9), we obtain [4] that ${}^{1/2}H(\{k_i\}) = H(\{k'_i\})$ where the set of parameters $\{k'_i\}$ is defined by the PS relation (8). Hence, the transformation $R(\pi/2)$ is equivalent to the PS transformation (8).

With the help of the transformation $R(\pi/2)$ we obtain a new monopole operator:

$$\overline{P} \equiv {}^{1/2}P = R(\pi/2)\, P = \frac{1}{2}\left((\tilde{d}\cdot\tilde{d}) + (s\cdot s)\right) . \tag{13}$$

This monopole operator corresponds to an alternative embedding of the SO(6) subalgebra in the U(6) algebra [8, 1]. Using the PS relation (8) it is easy to obtain [2, 4] the following expression for the Casimir operator of the $\overline{\mathrm{SO}(6)}$ algebra associated with the monopole operator $\overline{P}$:

$$C_2\left[\overline{\mathrm{SO}(6)}\right] = 10N + 4(N{+}2)\, C_1[\mathrm{U}(5)] - 4C_2[\mathrm{U}(5)] + 2C_2[\mathrm{SO}(5)] - C_2[\mathrm{SO}(6)] \, . \tag{14}$$

Note, that the Casimir operators of alternatively embedded algebras $\mathrm{SU}^1(3)$ and $\overline{\mathrm{SO}}(6)$ are not independent from the Casimir operators of other algebras and should not be included into the general Hamiltonian (3).

Transformations $\tilde{R}(\varphi)$ do not generate a new PS.

The Hamiltonian parameters $\{k_i\}$ are conventionally obtained by the fit to nuclear spectra. Due to PS, the fit of the parameters appears to be ambiguous. It seems natural to use electromagnetic transitions to discriminate between the isospectral parameter sets. However as it was shown in [4], the electromagnetic transitions rates cannot distinguish between PS-related Hamiltonians, at least within the so-called consistent-Q formalism [11] that is a conventional ansatz for the electromagnetic transition operator within IBM.

Nevertheless, in the case $k_6 = 0$ there is a possibility to discriminate between the Hamiltonians related by the PS (8). Since the relations (8) involve the total number of bosons N which is different for different nuclei, the two PS-related sets of the parameters can generate identical spectra for some particular nucleus only, the predictions for the spectra of neighboring nuclei should differ. Hence one can discriminate between the parameter sets $\{k_i\}$ and $\{k'_i\}$ related according to (8) by examining the spectra of a number of neighboring nuclei. This was illustrated in Ref. [4] by calculations of the spectra of Pt isotopes.

4 Parameter Symmetry of IBM-2

Within IBM-2 [1, 6], s and d bosons are introduced in the proton and neutron subspaces independently. The symmetry algebra of the model is $\mathrm{U}_\pi(6) \otimes \mathrm{U}_\nu(6)$ generated by 72 bilinear operators $s^+_\rho s_\rho$, $d^+_{\rho\mu} d_{\rho\nu}$, $d^+_{\rho\mu} s_\rho$, $s^+_\rho d_{\rho\mu}$ $(\rho = \pi,\, \nu)$.

The general IBM-2 Hamiltonian H consists of proton part H_π, neutron part H_ν and proton-neutron interaction $V_{\pi\nu}$,

$$H\left(\{k^\pi_i, k^\nu_i, k_i\}\right) = H_\pi\left(\{k^\pi_i\}\right) + H_\nu\left(\{k^\nu_i\}\right) + V_{\pi\nu}\left(\{k_i\}\right) , \tag{15}$$

and is characterized by 21 independent parameters $\{k^\pi_i, k^\nu_i, k_i\}$ [6]. The proton and neutron parts of the Hamiltonian, $H_\rho\left(\{k^\rho_i\}\right)$, $\rho = \pi, \nu$, are just the IBM-1 Hamiltonians and are given by (3) with $H_0 = 0$.

However, the expression of the proton-neutron interaction $V_{\pi\nu}\left(\{k_i\}\right)$ in terms of Casimir operators of combined proton-neutron subalgebras $\mathrm{G}_{\pi\nu}$ of the $\mathrm{U}_\pi(6)\otimes\mathrm{U}_\nu(6)$ algebra is unknown. The generators $\mathcal{G}_{\pi\nu}$ of the combined proton-neutron algebras $\mathrm{G}_{\pi\nu}$ are of the form $\mathcal{G}_{\pi\nu}=\mathcal{G}_\pi+\mathcal{G}_\nu$ where $\mathcal{G}_\pi$ and $\mathcal{G}_\nu$ are generators of the corresponding proton and neutron algebras, respectively.

The $\mathrm{U}_\pi(6)\otimes\mathrm{U}_\nu(6)$ algebra has a number of appropriate reduction chains. There are three types of reduction chains which include U(5), SU(3) and SO(6) subalgebras [1]:

1. U(5) DS reduction chains:

$$\begin{array}{ccccccc}
\mathrm{U}_\pi(6) & \supset & \mathrm{U}_\pi(5) & \supset & \mathrm{SO}_\pi(5) & \supset & \mathrm{SO}_\pi(3) \\
\cup & & \cup & & \cup & & \cup \\
\mathrm{U}_{\pi\nu}(6) & \supset & \mathrm{U}_{\pi\nu}(5) & \supset & \mathrm{SO}_{\pi\nu}(5) & \supset & \mathrm{SO}_{\pi\nu}(3) \\
\cup & & \cup & & \cup & & \\
\mathrm{U}_\nu(6) & \supset & \mathrm{U}_\nu(5) & \supset & \mathrm{SO}_\nu(5) & \supset & \mathrm{SO}_\nu(3)
\end{array} \tag{16}$$

2. SU(3) DS reduction chains:

$$\begin{array}{ccccccc}
\mathrm{U}_\pi(6) & \supset & \mathrm{SU}_\pi(3) & \supset & \mathrm{SO}_\pi(3) & & \\
\cup & & \cup & & \cup & & \\
\mathrm{U}_{\pi\nu}(6) & \supset & \mathrm{SU}_{\pi\nu}(3) & \supset & \mathrm{SO}_{\pi\nu}(3) & & \\
\cup & & \cup & & & & \\
\mathrm{U}_\nu(6) & \supset & \mathrm{SU}_\nu(3) & \supset & \mathrm{SO}_\nu(3) & &
\end{array} \tag{17}$$

3. SO(6) DS reduction chains:

$$\begin{array}{ccccccc}
\mathrm{U}_\pi(6) & \supset & \mathrm{SO}_\pi(6) & \supset & \mathrm{SO}_\pi(5) & \supset & \mathrm{SO}_\pi(3) \\
\cup & & \cup & & \cup & & \cup \\
\mathrm{U}_{\pi\nu}(6) & \supset & \mathrm{SO}_{\pi\nu}(6) & \supset & \mathrm{SO}_{\pi\nu}(5) & \supset & \mathrm{SO}_{\pi\nu}(3) \\
\cup & & \cup & & \cup & & \\
\mathrm{U}_\nu(6) & \supset & \mathrm{SO}_\nu(6) & \supset & \mathrm{SO}_\nu(5) & \supset & \mathrm{SO}_\nu(3)
\end{array} \tag{18}$$

The set of Casimir operators provided by the algebras entering the reduction chains (16)–(18), is not complete enough to express the general IBM-2 Hamiltonian (15). The only way to obtain this complete set is to look for alternative embeddings of the combined proton-neutron algebras. All the alternative subalgebras can be obtained by applying all possible transformations $R_\rho(\varphi_\rho)$ and $\tilde{R}_\rho(\varphi_\rho)$ to the generators of all subalgebras in the reduction chains (16)–(18). As a result, we obtain a large number of alternative subalgebras $\mathrm{G}_\rho^{\alpha_\rho}$ and $\mathrm{G}_{\pi\nu}^{\alpha_\pi\alpha_\nu}$ with generators ${}^{\alpha_\rho}\mathcal{G}_\rho$ and ${}^{\alpha_\pi\alpha_\nu}\mathcal{G}_{\pi\nu}={}^{\alpha_\pi}\mathcal{G}_\pi+{}^{\alpha_\nu}\mathcal{G}_\nu$, respectively. Some of the algebras $\mathrm{G}_{\pi\nu}^{\alpha_\pi\alpha_\nu}$ have been already introduced in literature. For example, the $\overline{\mathrm{SO}}_{\pi\nu}(6)$ algebra of Ref. [1] is the $\mathrm{SO}_{\pi\nu}^{1/2\,1/2}(6)$ algebra in our notations while the $\mathrm{SU}_{\pi\nu}^{0-1}(3)$ algebra is equivalent to the $\mathrm{SU}_{\pi\nu}^{*}(3)$ algebra introduced in Ref. [10] for the description of triaxial shapes within IBM-2. Following the book [1], we add the $\overline{\mathrm{SO}}(6)$ DS reduction chain to the reduction chains (16)–(18):

4. $\overline{\mathrm{SO}}(6)$ DS reduction chains:

$$\begin{array}{ccccccc}
\mathrm{U}_\pi(6) & \supset & \overline{\mathrm{SO}}_\pi(6) & \supset & \mathrm{SO}_\pi(5) & \supset & \mathrm{SO}_\pi(3) \\
\cup & & \cup & & \cup & & \cup \\
\mathrm{U}_{\pi\nu}(6) & \supset & \overline{\mathrm{SO}}_{\pi\nu}(6) & \supset & \mathrm{SO}_{\pi\nu}(5) & \supset & \mathrm{SO}_{\pi\nu}(3) \\
\cup & & \cup & & \cup & & \\
\mathrm{U}_\nu(6) & \supset & \overline{\mathrm{SO}}_\nu(6) & \supset & \mathrm{SO}_\nu(5) & \supset & \mathrm{SO}_\nu(3)
\end{array} \tag{19}$$

The reduction chains (16)–(19) will be referred to as standard DS reduction chains.

Some of the algebras $\mathrm{G}_{\pi\nu}^{\alpha_\pi\alpha_\nu}$ are equivalent. For example, any algebra $\mathrm{G}_{\pi\nu}^{\alpha_\pi\alpha_\nu}$ is equivalent to its proton-neutron particle-hole counterpart $\mathrm{G}_{\pi\nu}^{-\alpha_\pi-\alpha_\nu}$ — the relative sign of α_ν and α_π is only important, changing the sign of both α_ν and α_π we do not obtain a new algebra. As follows from our analysis, there exist 2 different realizations of $\mathrm{U}_{\pi\nu}(5)$, 2 different realizations of $\mathrm{SO}_{\pi\nu}(6)$, 2 different realizations of $\overline{\mathrm{SO}}_{\pi\nu}(6)$, and 8 different realizations of $\mathrm{SU}_{\pi\nu}(3)$.

The proton-neutron algebra $\overline{\mathrm{SO}}_{\pi\nu}(6)$ provides us with an additional Casimir operator $C_2[\overline{\mathrm{SO}}_{\pi\nu}(6)]$ that is independent from the Casimir operators of the groups entering the reduction chains (16)–(18) and can be used for the construction of $V_{\pi\nu}\left(\{k_i\}\right)$. However, the full set of the non-equivalent algebras $\mathrm{G}_\rho^{\alpha_\rho}$ and $\mathrm{G}_{\pi\nu}^{\alpha_\pi\alpha_\nu}$ provide us with an over-complete set of Casimir operators as some of the Casimir operators are not independent from the others. For example, the Casimir operators of alternative proton or neutron algebras $\mathrm{G}_\rho^{\alpha_\rho}$ can be expressed through the Casimir operators of untransformed algebras G_ρ [see Eqs. (12)

and (14)]; the rank-1 Casimir operator of any of $\mathrm{U}^{\alpha_\pi\alpha_\nu}_{\pi\nu}(5)$ algebras can be expressed through the Casimir operators of $\mathrm{U}^{\alpha_\pi}_{\pi}(5)$ and $\mathrm{U}^{\alpha_\nu}_{\nu}(5)$: $C_1[\mathrm{U}^{\alpha_\pi\alpha_\nu}_{\pi\nu}(5)] = C_1[\mathrm{U}^{\alpha_\pi}_{\pi}(5)] + C_1[\mathrm{U}^{\alpha_\nu}_{\nu}(5)]$, etc. Hence the choice of the complete set of independent Casimir operators is ambiguous.

The rank-2 Casimir operators of combined proton-neutron subalgebras $\mathrm{U}_{\pi\nu}(6)$, $\mathrm{U}_{\pi\nu}(5)$, $\mathrm{SO}_{\pi\nu}(6)$, $\overline{\mathrm{SO}}_{\pi\nu}(6)$, $\mathrm{SU}_{\pi\nu}(3)$, $\mathrm{SU}^{01}_{\pi\nu}(3)$ and $\mathrm{SU}^{11}_{\pi\nu}(3)$ form a complete set of independent operators that we propose to use for the construction of $V_{\pi\nu}(\{k_i\})$:

$$\begin{aligned} V_{\pi\nu}(\{k_i\}) = H_0 &+ kC_2[\mathrm{U}_{\pi\nu}(6)] + k_2C_2[\mathrm{U}_{\pi\nu}(5)] + k_3C_2[\mathrm{SO}_{\pi\nu}(5)] + k_4C_2[\mathrm{SO}_{\pi\nu}(3)] + k_5C_2[\mathrm{SO}_{\pi\nu}(6)] \\ &+ k_6C_2[\overline{\mathrm{SO}}_{\pi\nu}(6)] + k_7C_2[\mathrm{SU}_{\pi\nu}(3)] + k_8C_2[\mathrm{SU}^{01}_{\pi\nu}(3)] + k_9C_2[\mathrm{SU}^{11}_{\pi\nu}(3)]\,. \end{aligned} \tag{20}$$

The Casimir operator $C_2[\mathrm{U}(6)]$ not defined above can be expressed as

$$C_2\,[\mathrm{U}(6)] = (S\cdot S) + \frac{1}{2}(Q^0\cdot Q^0) + \frac{1}{2}(\overline{Q^0}\cdot\overline{Q^0}) + \sum_{\lambda=0}^{4}(T^\lambda\cdot T^\lambda)\,. \tag{21}$$

An alternative choice of independent Casimir operators was suggested in Ref. [12].

It is seen that the construction of different realizations of boson algebras plays an important role in IBM-2. The incompleteness of boson Hamiltonians in the form of superposition of Casimir invariants of different groups determined by standard DS reduction chains, is a common property of systems of two (or more) independent subsystems, e.g. it is also a property of the vibron model of triatomic molecules with the symmetry algebra $\mathrm{U}_1(4)\otimes\mathrm{U}_2(4)$ [13].

The standard reduction chains (16)–(19) define standard DS limits of IBM-2. As the Casimir operators of $\mathrm{SU}^{01}_{\pi\nu}$ and $\mathrm{SU}^{11}_{\pi\nu}$ are present in the Hamiltonian, we can also define non-standard $\mathrm{SU}^{01}_{\pi\nu}$ and $\mathrm{SU}^{11}_{\pi\nu}$ DS limits of IBM-2 that are associated with non-standard SU^{01} and SU^{11} DS reduction chains, respectively. The definition of non-standard DS limits of the model is, of course, ambiguous because of the ambiguity of definition of the complete set of independent Casimir operators.

Applying all possible transformations $R_\rho(\varphi_\rho)$ and $\tilde{R}_\rho(\varphi_\rho)$ to all subalgebras in the standard reduction chains (16)–(19), we obtain all alternative reduction chains. Some of these reduction chains appear to be equivalent to some of the others, some of them are equivalent to some of the standard or non-standard DS reduction chains. However the set of independent alternative reduction chains can be easily defined. These independent alternative reduction chains give rise to hidden symmetries of the model. It is interesting that some of the hidden symmetries may be obtained by means of transformations (e.g., by particle-hole transformations) that are not isospectral.

Applying all possible transformations $R_\rho(\varphi_\rho)$ and $\tilde{R}_\rho(\varphi_\rho)$ to the general Hamiltonian (15), we obtain a general IBM-2 Hamiltonian that can be (i) identical to the initial Hamiltonian (15), (ii) non-identical to but isospectral with the initial Hamiltonian (15), or (iii) non-isospectral with the initial Hamiltonian (15). In the case (ii) we obtain standard parameter symmetries of IBM-2 that are valid without restrictions on the parameters of the model. The standard parameter symmetries are listed in Table 2. In the case (iii) we do not immediately obtain PS. However, for any possible transformation $R_\rho(\varphi_\rho)$ [or $\tilde{R}_\rho(\varphi_\rho)$] there always can be found some constraints on the parameters $\{k^\pi_i, k^\nu_i, k_i\}$ such that the Hamiltonians $H(\{k^\pi_i, k^\nu_i, k_i\})$ and $R_\rho(\varphi_\rho)\,H(\{k^\pi_i, k^\nu_i, k_i\})$ [or $\tilde{R}_\rho(\varphi_\rho)\,H(\{k^\pi_i, k^\nu_i, k_i\})$] become isospectral even if there is no isospectrality between these Hamiltonians in the general case. Hence in the case (iii) we obtain additional non-standard parameter symmetries that are valid only if the parameters fit some relations. These additional symmetries are listed in Table 3 (the constraining relations for the parameters are given in the first row).

All IBM-2 PS relations with the only exception of the PS D (see Table 2), involve the total number of proton bosons N_π or/and the total number of neutron bosons N_ν ($N = N_\pi + N_\nu$). Hence there is a principal possibility to distinguish between few parameter sets giving rise to identical spectra by the analysis of the spectra of neighboring isotopes or/and isotones.

5 Summary

We have analyzed canonical transformations of boson operators consistent with the structure of boson Hamiltonian in the cases of IBM-1 and IBM-2, or, equivalently, different realizations of the symmetry

Table 2: Standard PS relations for the IBM-2 Hamiltonian

A	B	C	D
$R_\pi(0)\times R_\nu(0)$	$R_\pi(\pi)\times R_\nu(0)$	$R_\pi(0)\times R_\nu(\pi)$	$R_\pi(\pi)\times R_\nu(\pi)$
H_0	$H_0-10(k+2k_6)N$	$H_0-10(k+2k_6)N$	H_0
k_1^π	$k_1^\pi+2k_6^\pi+4(N_\pi+2)(2k_6+k)$	$k_1^\pi+4(N_\pi+2)(2k_6+k)$	$k_1^\pi+2k_6^\pi$
k_2^π	$k_2^\pi+2k_6^\pi-8k_6-4k$	$k_2^\pi-8k_6-4k$	$k_2^\pi+2k_6^\pi$
k_3^π	$k_3^\pi-6k_6^\pi-2k_5+2k_6-8k_7$	$k_3^\pi-2k_5+2k_6-8k_9$	$k_3^\pi-6k_6^\pi-8k_8$
k_4^π	$k_4^\pi+2k_6^\pi$	k_4^π	$k_4^\pi+2k_6^\pi$
k_5^π	$k_5^\pi+4k_6^\pi+2k_5-2k_6+8k_7$	$k_5^\pi+2k_5-2k_6+8k_9$	$k_5^\pi+4k_6^\pi+8k_8$
k_6^π	$-k_6^\pi$	k_6^π	$-k_6^\pi$
k_1^ν	$k_1^\nu+4(N_\nu+2)(k+2k_6)$	$k_1^\nu+2k_6^\nu+4(N_\nu+2)(k+2k_6)$	$k_1^\nu+2k_6^\nu$
k_2^ν	$k_2^\nu-8k_6-4k$	$k_2^\nu+2k_6^\nu-8k_6-4k$	$k_2^\nu+2k_6^\nu$
k_3^ν	$k_3^\nu-2k_5+2k_6-8k_7$	$k_3^\nu-6k_6^\nu-2k_5+2k_6-8k_9$	$k_3^\nu-6k_6^\nu-8k_8$
k_4^ν	k_4^ν	$k_4^\nu+2k_6^\nu$	$k_4^\nu+2k_6^\nu$
k_5^ν	$k_5^\nu+2k_5-2k_6+8k_7$	$k_5^\nu+4k_6^\nu+2k_5-2k_6+8k_9$	$k_5^\nu+4k_6^\nu+8k_8$
k_6^ν	k_6^ν	$-k_6^\nu$	$-k_6^\nu$
k	k	k	k
k_2	k_2	k_2	k_2
k_3	$k_3+2k_5+2k_6+2k+8k_7$	$k_3+2k_5+2k_6+2k+8k_9$	k_3+8k_8
k_4	k_4	k_4	k_4
k_5	$-k_5-k-8k_7$	$-k_5-k-8k_9$	k_5-8k_8
k_6	$-k_6-k$	$-k_6-k$	k_6
k_7	k_7	k_8+k_9	k_8+k_9
k_8	$-k_7+k_9$	k_7-k_9	$-k_8$
k_9	k_7+k_8	k_9	k_7+k_8

algebra of the model and its subalgebras. Analysis of alternatively embedded subalgebras is of a particular importance in the case of IBM-2 because it provides a regular way to construct the general IBM-2 Hamiltonian.

One can suppose that there should be no signals in physical applications of switching from one realization of the symmetry algebra to another equivalent realization. However, it is not so. The existence of alternative realizations of the symmetry algebra manifests itself as parameter symmetries of the model, i.e. as existence of few sets of Hamiltonian parameters providing identical spectra. The parameter symmetry is of physical importance for applications of IBM since the parameters of the model are obtained by the fit to experimental spectra. In some cases one can discriminate between the PS-related sets of parameters by analyzing spectra of neighboring nuclei.

The parameter symmetry is a common property of boson models. For example, we have shown [7] that it is present in the vibron model (see e.g. [1]); we suppose that it can be found in other algebraic models including fermion and boson-fermion ones. There can exist other possibilities of discriminating between isospectral parameter sets (see, e.g., Ref. [14]).

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Table 3: PS relations for the IBM-2 Hamiltonian with some constraints on the parameters

A	E : $\begin{cases} k{+}2k_5{+}8k_7=0 \\ k{+}2k_6=0 \\ k_9=k_7{+}k_8=0 \\ k_6^\pi=0 \end{cases}$	F : $\begin{cases} k{+}2k_5{+}8k_9=0 \\ k{+}2k_6=0 \\ k_7=k_8{+}k_9 \\ k_6^\nu=0 \end{cases}$	G : $\begin{cases} k_8=0 \\ k_7=k_9 \\ k_6^\pi=0 \\ k_6^\nu=0 \end{cases}$	H : $\begin{cases} k{+}2k_5{+}8k_7=0 \\ k{+}2k_6=0 \\ k_9=k_7{+}k_8 \\ k_6^\pi=0 \end{cases}$	I : $\begin{cases} k{+}2k_5{+}8k_9=0 \\ k{+}2k_6=0 \\ k_7=k_8{+}k_9 \\ k_6^\nu=0 \end{cases}$
	$R_\pi(\pi/2)\times R_\nu(0)$	$R_\pi(0)\times R_\nu(\pi/2)$	$R_\pi(\pi/2)\times R_\nu(\pi/2)$	$R_\pi(\pi/2)\times R_\nu(\pi)$	$R_\pi(\pi)\times R_\nu(\pi/2)$
H_0	$H_0-10(k_5^\pi{+}4k_8)N_\pi$	$H_0-10(k_5^\nu{+}4k_8)N_\nu$	$H_0-10(k_5^\pi N_\pi-k_5^\nu N_\nu)$	$H_0-10(k_5^\pi{+}4k_8)N_\pi$	$H_0-10(k_5^\nu{+}4k_8)N_\nu$
k_1^π	$k_1^\pi{+}4(N_\pi{+}2)(k_5^\pi{+}4k_8)$	k_1^π	$k_1^\pi{+}4(N_\pi{+}2)k_5^\pi$	$k_1^\pi{+}4(N_\pi{+}2)(k_5^\pi{+}4k_8)$	$k_1^\pi{+}2k_7^\pi$
k_2^π	$k_2^\pi{-}4k_5^\pi{-}16k_8$	k_2^π	$k_2^\pi{-}4k_5^\pi$	$k_2^\pi{-}4k_5^\pi{-}16k_6$	$k_2^\pi{+}2k_6^\pi$
k_3^π	$k_3^\pi{+}2k_5^\pi{+}8k_8$	k_3^π	$k_3^\pi{+}2k_5^\pi$	$k_3^\pi{+}2k_5^\pi$	$k_3^\pi{-}6k_6^\pi{-}8k_8$
k_4^π	k_4^π	k_4^π	k_4^π	k_4^π	$k_4^\pi{+}2k_6^\pi$
k_5^π	$-k_5^\pi{-}8k_8$	k_5^π	$-k_5^\pi$	$-k_5^\pi$	$k_5^\pi{+}4k_6^\pi{+}8k_8$
k_6^π	$k_6^\pi=0$	k_6^π	$k_6^\pi=0$	$k_6^\pi=0$	$-k_6^\pi$
k_1^ν	k_1^ν	$k_1^\nu{+}4(N_\nu{+}2)(k_5^\nu{+}4k_8)$	$k_1^\nu{+}4(N_\nu{+}2)k_5^\nu$	$k_1^\nu{+}2k_6^\nu$	$k_1^\nu{+}4(N_\nu{+}2)(k_5^\nu{+}4k_8)$
k_2^ν	k_2^ν	$k_2^\nu{-}4k_5^\nu{-}16k_8)$	$k_2^\nu{-}4k_5^\nu$	$k_2^\nu{+}2k_6^\nu$	$k_2^\nu{-}4k_5^\nu{-}16k_8$
k_3^ν	k_3^ν	$k_3^\nu{+}2k_5^\nu{+}8k_8$	$k_3^\nu{+}2k_5^\nu$	$k_3^\nu{-}6k_6^\nu{-}8k_8$	$k_3^\nu{+}2k_5^\nu$
k_4^ν	k_4^ν	k_4^ν	k_4^ν	$k_4^\nu{+}2k_6^\nu$	k_4^ν
k_5^ν	k_5^ν	$-k_5^\nu{-}8k_8$	$-k_5^\nu$	$k_5^\nu{+}4k_6^\nu{+}8k_8$	$-k_5^\nu$
k_6^ν	k_6^ν	$k_6^\nu=0$	$k_6^\nu=0$	$-k_6^\nu$	$k_6^\nu=0$
k	k	k	k	k	k
k_2	k_2	k_2	k_2	k_2	k_2
k_3	k_3	k_3	k_3	$k_3{+}8k_8$	$k_3{+}8k_8$
k_4	k_4	k_4	k_4	k_4	k_4
k_5	k_5	k_5	$k_6{-}4k_7$	$k_5{-}8k_8$	$k_5{-}8k_8$
k_6	k_6	k_6	$k_5{+}4k_7$	k_6	k_6
k_7	k_7	k_7	k_7	$k_8{+}k_9$	k_7
k_8	k_8	k_8	$k_8=0$	$-k_8$	$-k_8$
k_9	k_9	k_9	k_9	k_9	$k_7{+}k_8$

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