Higher Spin Currents in Arbitrary Dimension

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Abstract

We construct a new class of higher spin currents in arbitrary dimension which generalize angular-momentum current.

Consistent theories of massless fields of arbitrary spin result from gauging specific symmetries called higher spin symmetries. Higher spin symmetries therefore play a fundamental role in the higher spin gauge theory. Since the Noether theorem links together symmetries and conserved currents it is useful to analyse possible conserved currents to work out a structure of higher spin symmetries in arbitrary space-time dimension, which by now are only known in d=2,3 and 4 (see [1] and references therein). This is the main motivation for the analysis of higher spin conserved currents in this contribution.

To begin with let us recall that usual inner symmetries are related via the Noether theorem to the conserved spin 1 current that can be constructed from different matter fields. For example, a current constructed from scalar fields in an appropriate representation of the gauge group

$$J^{\underline{n}}{}_i{}^j = \bar{\phi}_i \partial^{\underline{n}} \phi^j - \partial^{\underline{n}} \bar{\phi}_i \phi^j \tag{1}$$

is conserved on the solutions of the scalar field equations

$$\partial_{\underline{n}} J^{\underline{n}j}_i = \bar{\phi}_i (\Box + m^2)\phi^j - (\Box + m^2)\bar{\phi}_i \phi^j \,. \tag{2}$$

(Underlined indices are used for differential forms and vector fields in d-dimensional space-time, i.e. $\underline{n} = 0, \ldots, d-1$ while i and j are inner indices.)

Translational symmetry is associated with the spin 2 current called stress tensor. For scalar matter it has the form

$$T^{\underline{mn}} = \partial^{\underline{m}}\phi \partial^{\underline{n}}\phi - \frac{1}{2}\eta^{\underline{mn}} \left(\partial_{\underline{r}}\phi \partial^{\underline{r}}\phi - m^2\phi^2 \right) \,. \tag{3}$$

Supersymmetry is based on the conserved current called supercurrent. It has fermionic statistics and is constructed from bosons and fermions. For massless scalar ϕ and massless spinor ψ_ν it has the form

$$J^{\underline{n}}{}_\nu = \partial_{\underline{m}}\phi (\gamma^{\underline{m}}\gamma^{\underline{n}}\psi)_\nu \,, \tag{4}$$

where $\gamma^n{}_\mu{}^\nu$ are Dirac matrices in d dimensions.

The conserved charges, associated with these conserved currents, correspond, respectively, to generators of inner symmetries $T^i{}_j$, space-time translations P^n and supertransformations Q^i_ν. The conserved current associated with Lorentz rotations can be constructed from the symmetric stress tensor

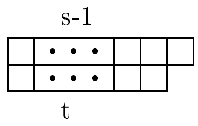
$$S^{\underline{n}}{}_{;}{}^{ml} = T^{\underline{nm}} x^{\underline{l}} - T^{\underline{nl}} x^{\underline{m}} \,, \qquad T^{\underline{nm}} = T^{\underline{mn}} \,. \tag{5}$$

These exhaust the standard lower spin conserved currents usually used in the field theory.

The list of lower spin currents admits a natural extension to higher spin currents containing higher derivatives of the physical fields. The higher spin currents associated with the integer spin s

$$J^{\underline{n}}{}_{;m_1 \ldots m_t, n_1 \ldots n_{s-1}} \tag{6}$$

are vector fields (index $\underline{n}$) taking values in all representations of the Lorentz group described by the traceless two-row Young diagrams



(7)

with $0 \le t \le s-1$. This means that the currents $J^{\underline{n};m_1\dots m_t,n_1\dots n_{s-1}}$ are symmetric in the indices n and m, satisfy the relations

$$(s-1)(s-2)J^{\underline{n}}{}_{;m_1\dots m_t\,,r}{}^{r}{}_{n_3\dots n_{s-1}} = 0\,, \tag{8}$$

$$t(s-1)J^{\underline{n}}{}_{;rm_2\dots m_t\,,}{}^{r}{}_{n_2\dots n_{s-1}}\,, \qquad t(t-1)J^{\underline{n}}{}_{;m_3\dots m_t r}{}^{r}{}_{,n_1\dots n_{s-1}} = 0\,, \tag{9}$$

and obey the antisymmetry property

$$J^{\underline{n}}{}_{;m_2\dots m_t\{n_s\,,n_1\dots n_{s-1}\}_n} = 0\,, \tag{10}$$

implying that symetrization over any s indices n and/or m gives zero.

Let us now explain notation, which simplifies analysis of complicated tensor structures and is useful in the component analysis. Following [2] we combine the Einstein rule that upper and lower indices denoted by the same letter are to be contracted with the convention that upper (lower) indices denoted by the same letter imply symmetrization which should be carried out prior contractions. With this notation it is enough to put a number of symmetrized indices in brackets writing e.g. $X_{n(p)}$ instead of $X_{n_1\dots n_p}$.

Now, the higher spin currents are $J^{\underline{n}}{}_{;m(t)\,,n(s-1)}$ $(1 \le t \le s-1)$ while the conditions (8)-(10) take the form

$$J^{\underline{n}}{}_{;m(t)\,,n(s-2)}{}^{n} = 0\,, \quad J^{\underline{n}}{}_{;m(t-1)}{}^{n}{}_{,n(s-1)} = 0\,, \quad J^{\underline{n}}{}_{;m(t)}{}^{m}{}_{,n(s-1)} = 0\,, \tag{11}$$

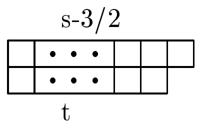
and

$$J^{\underline{n}}{}_{;m(t-1)n\,,n(s-1)} = 0\,. \tag{12}$$

The higher spin supercurrents associated with half-integer spins

$$J^{\underline{n}}{}_{;m_1\dots m_t,n_1\dots n_{s-3/2};\nu} \tag{13}$$

are vector fields (index $\underline{n}$) taking values in all representations of the Lorentz group described by the γ -transversal two-row Young diagrams



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i.e., the irreducibility conditions for the higher spin supercurrents $J^{\underline{n}}{}_{;m(t)\,,n(s-3/2);\nu}$ read

$$tJ^{\underline{n}}{}_{;m(t-1)n,n(s-3/2);\nu} = 0 \tag{15}$$

and

$$(s-5/2)\,\gamma^{n}{}_{\mu}{}^{\nu}J^{\underline{n}}{}_{;m(t),n(s-3/2);\nu} = 0\,. \tag{16}$$

From these conditions it follows that

$$(s-5/2)\gamma^{m}{}_{\mu}{}^{\nu}J^{\underline{n}}{}_{;m(t),n(s-3/2);\nu} = 0 \tag{17}$$

and all tracelessness conditions (8) and (9) are satisfied.

To avoid complications resulting from the projection to the space of irreducible (i.e. traceless or $\gamma-$ transversal) two–row Young diagrams we study the currents

$$J^{\underline{n}}(\xi) = J^{\underline{n}}{}_{;}{}^{m(t)}{}_{,}{}^{n(s-1)}\xi_{m(t),n(s-1)}\,, \quad J^{\underline{n}}(\xi) = \xi_{m(t),n(s-3/2);}{}^{\nu}J^{\underline{n}}{}_{;}{}^{m(t)}{}_{,}{}^{n(s-3/2)}{}_{;\nu}\,, \tag{18}$$

where $\xi_{m(t),n(s-1)}$ and $\xi_{m(t),n(s-3/2);}{}^{\nu}$ are some constant parameters which themselves satisfy analogous irreducibility conditions. The conservation law then reads

$$\partial_{\underline{n}}J^{\underline{n}}(\xi) = 0\,. \tag{19}$$

The currents corresponding to one-row Young diagrams (i.e. with $t=0$) generalize the spin 1 current (1), supercurrent (4) and stress tensor (3). An important fact is that they can be chosen in the form

$$J^{m}{}_{;\,,}{}^{n(s-1)}\xi_{n(s-1)} = T^{m\,n(s-1)}\xi_{n(s-1)} \tag{20}$$

$$J^{m}{}_{;\,,}{}^{n(s-3/2)}{}_{;\nu}\xi_{n(s-3/2);}{}^{\nu} = T^{m\,n(s-3/2)}{}_{;\nu}\xi_{n(s-3/2);}{}^{\nu}\,, \tag{21}$$

with totally symmetric conserved currents $T^{n(s)}$ or supercurrents $T^{n(s-1/2)}{}_{;\nu}$,

$$\xi_{n(s-1)}\partial_n T^{n(s)} = 0\,, \qquad \xi_{n(s-3/2);}{}^{\nu}\partial_n T^{n(s-1/2)}{}_{;\nu} = 0 \tag{22}$$

($\xi^{n}{}_{n(s-2)} = 0$, $(\xi_{n(s-3/2)}\gamma^n)^\nu = 0$).

Analogously to the formula (5) for the angular momenta current, the symmetric (super)currents T allow one to construct explicitly x-dependent higher spin "angular" currents. An observation is that the angular higher spin (super)currents

$$J^{\underline{n}}(\xi) = T^{\underline{n}n(s-1)}x^{m(t)}\xi_{m(t),n(s-1)}\,, \quad J^{\underline{n}}(\xi) = T^{\underline{n}\,n(s-3/2)}{}_{;\nu}x^{m(t)}\xi_{m(t),n(s-3/2);}{}^{\nu}\,, \tag{23}$$

where we use the shorthand notation

$$x^{m(s)} = \underbrace{x^m\ldots x^m}_{s}\,, \tag{24}$$

also conserve as a consequence of (22) because when the derivative in (19) hits a factor of x^m, the result vanishes by symmetrization of too many indices in the parameters ξ forming the two-row Young diagrams.

Since the parameters $\xi_{m(t),n(s-1)}$ and $\xi_{m(t),n(s-3/2);}{}^{\nu}$ are traceless and $\gamma-$transversal, only the double traceless part of $T^{\underline{n}n(s-1)}$

$$T^{n(2)}{}_{n(s-2)} = 0\,, \qquad s\geq 4 \tag{25}$$

and triple $\gamma-$transversal part of $T^{\underline{n}\,n(s-3/2)}{}_{;\nu}$

$$\gamma^n T^{n}{}_{n(s-3/2)} = 0\,, \qquad s\geq 7/2\,. \tag{26}$$

contribute to (23). These are the (super)currents of the formalism of symmetric tensors (tensor-spinors) [3, 4]. Some currents with integer spins $T^{n(s)}$ were considered in [5] for the particular case of massless matter fields.

Integer spin currents built from scalars of equal masses

$$(\Box + m^2)\phi^i = 0 \tag{27}$$

have the form

$$\begin{aligned} T^{n(2k)ij} = \;& (\partial^{n(k)}\phi^i\partial^{n(k)}\phi^j - \frac{k}{2}\eta^{nn}\partial^{n(k-1)}\partial_m\phi^i\partial^{n(k-1)}\partial^m\phi^j \\ & + \frac{k}{2}m^2\eta^{nn}\partial^{n(k-1)}\phi^i\partial^{n(k-1)}\phi^j) + i\leftrightarrow j \end{aligned} \tag{28}$$

for even spins and

$$T^{n(2k+1)ij} = (\partial^{n(k+1)}\phi^i\partial^{n(k)}\phi^j) - i\leftrightarrow j \tag{29}$$

for odd spins, where we use notation analogous to (24)

$$\partial^{n(s)} = \underbrace{\partial^n\ldots\partial^n}_{s}\,. \tag{30}$$

Higher spin supercurrents built from scalar and spinor with equal masses

$$(\Box + m^2)\phi = 0\,, \qquad (i\partial_{\underline{n}}\gamma^{\underline{n}} + m)\psi_\nu = 0 \tag{31}$$

read

$$T^{n(k+1)}{}_{;\nu} = \partial^{n(k+1)}\phi\psi_\nu - \frac{k+1}{2}\Big((\gamma^n\gamma^m\psi)_\nu\partial^{n(k)}\partial_m\phi + im(\gamma^n\psi)_\nu\partial^{n(k)}\phi\Big)\,. \tag{32}$$

Inserting these expressions into (23) we obtain the set of conserved "angular" higher spin currents of even, odd and half-integer spins. The usual angular momentum current corresponds to the case $s = 2$, $t = 1$.

Remarkably, the conserved higher spin currents listed above are in one–to–one correspondence with the higher spin gauge fields (1-forms) $\omega_{\underline{n};m(t),n(s-1)}$ and $\omega_{\underline{n};m(t),n(s-3/2);\nu}$ introduced for the free boson [6] and fermion [7] cases in arbitrary d. Of course, such a correspondence is expected because, like the gauge fields of the supergravitational multiplets, the higher spin gauge fields should take their values in a (infinite-dimensional) higher spin algebra identified with the global symmetry algebra in the corresponding dynamical system (this fact is explicitly demonstrated below for the cases of $d = 3$ and $d = 4$). The higher spin currents can then be derived via the Noether theorem from the global higher spin symmetry and give rise to the conserved charges identified with the Hamiltonian generators of the same symmetries.

A few comments are now in order.

Higher spin currents contain higher derivatives. Therefore, higher spin symmetries imply, via the Noether procedure, the appearance of higher derivatives in interactions.

Nontrivial (interacting) theories exhibiting higher spin symmetries are formulated in AdS background rather than in the flat space. Therefore an important problem is to generalize the constructed currents to the AdS geometry. This problem was solved recently [8] for the case d=3.

Explicit form of the higher spin algebras is known for $d \le 4$ [1] (and references therein) although a conjecture was made in [7] on the structure of higher spin symmetries in any d. The knowledge of the structure of the higher spin currents in arbitrary dimension may be very useful for elucidating a structure of the higher spin symmetries in any d.

Let us now compare the structure of higher spin currents with that of higher spin algebra in $d = 3$ and $d = 4$. The structure of the d3 higher spin gauge fields [9] suggests that the d3 higher spin currents should have a structure

$$J^{\underline{n};B\,,\alpha_1\ldots\alpha_n}(x)\,, \tag{33}$$

where, $B = 0,1$ and $\alpha = 1,2$ is spinor index. In other words, d=3 higher spin currents constitute two sets of multispinors of arbitrary rank. The label B distinguishes between the Lorentz–type fields ($B = 0$) and frame–type fields ($B = 1$) and therefore leads to two different types of currents of any spin $s = \frac{1}{2}n$.

To see that such a structure of currents fits the systematics of the higher spin currents one observes that in three dimensions all traceless two-row Young diagrams (7) with $t > 1$ vanish. Indeed, consider the quantity

$$Y^{cc}{}_{b(t),a(p)} = \epsilon^{abc}\epsilon^{abc}X_{b(t+2),a(p+2)} \tag{34}$$

with some traceless $X_{b(t+2),a(p+2)}$ satisfying the antisymmetry condition $X_{b(t+1)a,a(p+2)} = 0$. On the one hand, one gets

$$X_{b(t+2),a(p+2)} = \frac{p+1}{p+3}\epsilon_{abc}\epsilon_{abc}Y^{cc}{}_{b(t),a(p)}\,. \tag{35}$$

But on the other hand, from (34) it follows that $Y^{cc}{}_{b(t),a(p)} = 0$ since at least one pair of indices of $X_{b(t+2),a(p+2)}$ will be contracted after the two epsilon symbols are replaced by the combinations of the metric tensors.

As a result, only the currents corresponding to the Young diagrams (7) with $t = 0$ and $t = 1$ survive in the d3 case. The one-row currents ($t = 0$) describe stress tensor-type higher spin currents and correspond to the currents (33) with $B = 1$ while those with a single box in the second row describe higher spin angular momentum–type currents and correspond to the currents (33) with $B = 0$.

In $d = 4$, the full set of currents is described by all types of spinors [10]

$$J^{\underline{n}}{}_{;\alpha_1\ldots\alpha_n\,,\dot\alpha_1\ldots\dot\alpha_m}(x) \tag{36}$$

($\alpha = 1,2$, $\dot\alpha = 1,2$ are two-component spinor indices). It indeed describes in terms of two-component spinors the set of all two-row Young diagrams (7) both in the integer spin case ($n+m$ is even) and in the half-integer spin case ($n+m$ is odd), with the identification

$$s = \frac{1}{2}(n+m) + 1\,, \qquad t = \left[\frac{1}{2}|n-m|\right]\,, \tag{37}$$

where $[a]$ denotes integer part of a.

The fact that d3 and d4 higher spin algebras give rise to the sets of gauge fields which exactly match the sets of conserved higher spin currents described in this contribution is very significant. The derived structure of higher spin currents is a good starting point for the analysis of higher spin symmetries in higher dimensions.

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