Bogolyubov Group Variables for Poincaré-invariant Systems

O. A. Khrustalev[1], **M. V. Tchitchikina**[2]
Bogolyubov Institute for Theoretical Problems of Microphysics, Moscow State University,
Department of Physics, Moscow State University,
Moscow, 119899, Russia

and

S. Yu. Vernov[3]
Skobeltsyn Institute of Nuclear Physics, Moscow State University,
Moscow, 119899, Russia

Abstract

The $(3+1)$–dimensional Poincaré-invariant systems are considered. In the terms of Bogolyubov group variables procedure for quantization close to nonstationary classical fields has been developed. Use of Bogolyubov group variables allows to combine account of the conservation laws and perturbation theory and solve the zero-mode problem.

1 Introduction

The purpose of the present paper is to consider the problem of quantization close to nonstationary classical fields. It is well-known that careless introduction of nontrivial classical fields leads to violation of the conservation laws and to the problem of the so-called zero modes, which reflects incompatibility between the adopted perturbation theory and the symmetry properties of the system.

For a system with adiabatic coupling the problem of combining of the conservation laws with perturbation theory has been solved by N.N. Bogolyubov in 1950, when he studied the polaron problem [1]. The further investigations have shown that Bogolyubov group variables have a wide range of application. These variables have been used in the strong coupling theory [2] and other fields of the theoretical physics [3–7]. The Bogolyubov method has been independently rediscovered in the middle of 1970s and named collective coordinate method (see [6, 8] and references therein). Connections between various variants of this method have been discussed in [3].

Bogolyubov group variables are some operators, which can be treated as parameters of the system symmetry group. As an example let's consider a system with translational symmetry. After Bogolyubov transformation a static classical solution $\sigma(x)$ transforms in $\sigma(x-\hat{a})$, where $\hat{a}$ is an operator, canonical conjugate to an operator of the total linear momentum. Group variables are cyclic due to Hamiltonian symmetry, so the conservation laws are accomplished. It was revealed however that consideration of systems with a nonstationary classical components is a rather nontrivial problem, because explicit Hamiltonian structure as the time-translation generator becomes clear only after solving the motion equations. So we have to construct such Bogolyubov's scheme that group variables are defined in accordance with perturbative scheme, which is specified together with calculations of integrals of motion.

In the paper [8] Bogolyubov group variables have been defined for the $(1+1)$–dimensional Poincaré-invariant systems. Use of these variables allows, for example, to quantize a system with a nonstationary polaron [9]. In the present report we consider the $(3+1)$–dimensional Poincaré-invariant systems.

2 Bogolyubov transformation

We investigate the scalar theory with the following Lagrangian density:

$$\mathcal{L} = \frac{1}{2} g^{\alpha\beta} f_{,\alpha}(\mathbf{x}) f_{,\beta}(\mathbf{x}) - G^2 V\left(\frac{f}{G}\right), \tag{1}$$

[1]E-mail address: khrust@sunny.bog.msu.su
[2]E-mail address: chich@srdlan.npi.msu.su
[3]E-mail address: svernov@theory.npi.msu.su

where

$$\mathbf{x} = (x^0, x^1, x^2, x^3) \equiv (t, x)\,; \qquad f_{,\alpha} \equiv \frac{\partial f}{\partial x^\alpha}\,; \qquad g^{\alpha\beta} = diag(1, -1, -1, -1)\,; \qquad G \gg 1.$$

The Lagrange–Euler equation is

$$g^{\alpha\beta}\frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} + GV'\left(\frac{f}{G}\right) = 0. \tag{2}$$

This equation is invariant under the Poincaré group. Let $\tilde{\tau} \equiv (\tau^0, \tau^1, \ldots, \tau^9)$ denote parameters of Poincaré group. The Poincaré transformed coordinates $\mathbf{x}'$ are connected with the initial ones $\mathbf{x}$ by the following formula:

$$x'^\alpha = L^\alpha_\beta(\tau^4, \ldots, \tau^9)x^\beta + \tau^\beta, \tag{3}$$

where L^α_β is the matrix of the Lorentz transformation.

Let f be a field function, including a fixed classical component v and a quantum component u. Performing *the Bogolyubov transformation*:

$$f(\mathbf{x}') = G \cdot v(\mathbf{x}(\mathbf{x}', \tilde{\tau})) + u(\mathbf{x}(\mathbf{x}', \tilde{\tau})) \equiv G \cdot v(\mathbf{x}', \tilde{\tau}) + u(\mathbf{x}', \tilde{\tau}), \qquad G \gg 1, \tag{4}$$

we separate these components and treat v and u as functions of independent variables $\{\mathbf{x}', \tilde{\tau}\}$.[4]

A variation of $v(\mathbf{x}', \tilde{\tau})$ can be expressed in terms of linear independent variations $\delta\tau^a$:

$$\delta v(\mathbf{x}', \tilde{\tau}) \equiv \delta v(\mathbf{x}(\mathbf{x}', \tilde{\tau})) = \frac{\partial v}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial \tau^c}\delta\tau^c \equiv -M_c(\mathbf{x}', \tilde{\tau})\delta\tau^c. \tag{5}$$

Now there are 10 additional degrees of freedom, because the parameters of transformations τ^b are new independent variables, hence, we have to superpose ten subsidiary independent conditions on the function $u(\mathbf{x}', \tilde{\tau})$.

With the aid of some equation in $\mathbf{x}'$ one can define a space-like huperplane C and determine a bilinear functional

$$\omega(g_1, g_2) \equiv \int\limits_C \Big(g_{1n}(\lambda')g_2(\lambda') - g_1(\lambda')g_{2n}(\lambda')\Big)\, dS, \tag{6}$$

where g_1 and g_2 are differentiable functions, the operator $\frac{\partial}{\partial \mathbf{n}} \equiv n'^\alpha \frac{\partial}{\partial x'^\alpha}$ is a normal derivative, $dS \equiv d\lambda'^1 \ldots d\lambda'^{K-1}$. The operator $\frac{\partial}{\partial \mathbf{n}}$ is invariant under Poincaré group transformations. If $\mathbf{x}' \in C$, then functions $g(\mathbf{x}(\mathbf{x}', \tilde{\tau}))$ will be denoted as $g(\lambda')$.

To reduce the number of degrees of freedom we demand that the function u has to satisfy the following conditions:

$$\forall a = 0..9 \quad : \quad \omega(\tilde{N}^a, u) = 0, \tag{7}$$

where $\tilde{N}^a(\mathbf{x}', \tilde{\tau})$ are such differentiable functions that

$$\forall a, b = 0..9 \quad : \quad \omega(\tilde{N}^a, M_b) = \delta^a_b, \qquad \omega(\tilde{N}^a, \tilde{N}^b) = 0. \tag{8}$$

It is easy to show, that function u, satisfying conditions (7), can be constructed from an arbitrary differentiable function y by the following formula:

$$u(\mathbf{x}', \tilde{\tau}) = \hat{K}y(\mathbf{x}', \tilde{\tau}) \equiv y(\mathbf{x}', \tilde{\tau}) - M_a(\mathbf{x}', \tilde{\tau}) \cdot \omega(\tilde{N}^a, y), \tag{9}$$

where the operator $\hat{K}$ is a projecting operator: $\hat{K}^2 = \hat{K}$. Hence, conditions (7) define a subspace of the space of differentiable functions, which consists of solutions of the equation $u = \hat{K}u$.

Since subsidiary conditions (7) have been imposed on the functions $u(\lambda')$ and $u_n(\lambda')$, we cannot consider differentiation over them as differentiation over independent functions. As a result of this, the variational derivatives $\frac{\delta}{\delta u(\lambda')}$ and $\frac{\delta}{\delta u_n(\lambda')}$ are connected by the following relations [4, 8]:

$$\int\limits_C \left(M_a(\lambda')\frac{\delta}{\delta u(\lambda')} + M_{an}(\lambda')\frac{\delta}{\delta u_n(\lambda')}\right) dS = 0. \tag{10}$$

[4] All functions of argument $\mathbf{x}(\mathbf{x}', \tilde{\tau})$ are denoted as functions of $\mathbf{x}'$, $\tilde{\tau}$.

After the substitution $u(\lambda') \equiv u(\mathbf{x}(\mathbf{x}',\tilde{\tau})) = f(\mathbf{x}') - Gv(\lambda')$ in formulas (7), we obtain, that the consequences of invariance of subsidiary conditions (7) with respect to changes of τ^a are the following equations[5]:

$$\int\limits_C \left(\tilde{N}^a(\lambda')\delta f_n(\mathbf{x}') - \tilde{N}^a_n(\lambda')\delta f(\mathbf{x}')\right) dS - R^a_b\delta\tau^b - G\delta\tau^a = 0, \tag{11}$$

where

$$R^a_b \equiv \int\limits_C \left(\tilde{N}^a_{bn}(\lambda')u(\lambda') - \tilde{N}^a_b(\lambda')u_n(\lambda')\right) dS, \qquad \delta\tilde{N}^a(\mathbf{x}',\tilde{\tau}) \equiv \tilde{N}^a_b(\mathbf{x}',\tilde{\tau})\delta\tau^b.$$

These relations can be formulated in the following form:

$$\frac{\delta\tau^a}{\delta f(\mathbf{x}')} = -\frac{1}{G}Q^a_b\tilde{N}^b_n(\lambda'), \qquad \frac{\delta\tau^a}{\delta f_n(\mathbf{x}')} = \frac{1}{G}Q^a_b\tilde{N}^b(\lambda'), \tag{12}$$

where Q^a_b are solutions of the following system of equations:

$$Q^a_b = \delta^a_b - \frac{1}{G}R^a_cQ^c_b \qquad \Longrightarrow \qquad Q^a_b = \delta^b_a - \frac{1}{G}R^a_b + \frac{1}{G^2}R^a_cR^c_b + \mathcal{O}(G^{-3}).$$

Using formula (10) and obtained expressions for $\frac{\delta\tau^a}{\delta f(\mathbf{x}')}$ and $\frac{\delta\tau^a}{\delta f_n(\mathbf{x}')}$, we deduce:

$$\frac{\delta}{\delta f(\mathbf{x}')} = \frac{\delta}{\delta u(\lambda')} + \frac{\delta\tau^a}{\delta f(\mathbf{x}')}\left(\frac{\partial}{\partial\tau^a} + \hat{S}_a\right), \qquad \frac{\delta}{\delta f_n(\mathbf{x}')} = \frac{\delta}{\delta u_n(\lambda')} + \frac{\delta\tau^a}{\delta f_n(\mathbf{x}')}\left(\frac{\partial}{\partial\tau^a} + \hat{S}_a\right), \tag{13}$$

where

$$\hat{S}_a = \int\limits_C \left(u_a(\lambda')\frac{\delta}{\delta u(\lambda')} + u_{an}(\lambda')\frac{\delta}{\delta u_n(\lambda')}\right) dS, \qquad u_a(\lambda') \equiv \frac{\partial u(\lambda')}{\partial\tau^a}.$$

3 Quantization in terms of Bogolyubov group variables

3.1 Operators of coordinate and linear momentum

The operators

$$\hat{q}(\mathbf{x}') = \frac{1}{\sqrt{2}}\left(f(\mathbf{x}') + i\frac{\delta}{\delta f_n(\mathbf{x}')}\right) \qquad \text{and} \qquad \hat{p}(\mathbf{x}') = \frac{1}{\sqrt{2}}\left(f_n(\mathbf{x}') - i\frac{\delta}{\delta f(\mathbf{x}')}\right)$$

are defined on the space of the functionals $\Phi[f, f_n]$, with the following scalar production:

$$<\Phi_1|\Phi_2> = \int DfDf_n\Phi_1^*[f,f_n]\Phi_2[f,f_n].$$

The operators $\hat{q}(\mathbf{x}')$ and $\hat{p}(\mathbf{x}')$ are self-conjugate ones and satisfy the commutation relations:

$$\forall \mathbf{x}', \mathbf{x}'' \in C \quad : \quad [\hat{q}(\mathbf{x}'), \hat{p}(\mathbf{x}'')] = i\delta(\mathbf{x}' - \mathbf{x}''). \tag{14}$$

So, we can treat the operators $\hat{q}(\mathbf{x}')$ and $\hat{p}(\mathbf{x}')$ as operators of coordinate and linear momentum and develop the secondary quantization scheme. But straightforward use of this procedure leads to doubling the number of possible field states. We consider the functions $f(\mathbf{x}')$ and $f_n(\mathbf{x}')$ as independent functions, so, the self-conjugate operators

$$\tilde{q}(\mathbf{x}') = \frac{1}{\sqrt{2}}\left(f_n(\mathbf{x}') + i\frac{\delta}{\delta f(\mathbf{x}')}\right) \qquad \text{and} \qquad \tilde{p}(\mathbf{x}') = \frac{1}{\sqrt{2}}\left(f(\mathbf{x}') - i\frac{\delta}{\delta f_n(\mathbf{x}')}\right)$$

[5]From this point on we shall consider the functions $f(\mathbf{x}')$ and $f_n(\mathbf{x}')$ as independent ones.

satisfy the standard commutation relation (14) and commutate with the operators $\hat{q}$ and $\hat{p}$. To reduce the numbers of field states we can use, for example, the holomorphic representation, such procedure was described in detail in the articles [4, 8].

But after reduction of field states number it is difficult to separate the field function on classical and quantum parts and satisfy the subsidiary conditions. So, we shall develop the perturbation theory in spite of appearance of superfluous states, and only after this we shall make reduction of field states number.

The operators $\hat{q}$ and $\hat{p}$ include the operators $\frac{\partial}{\partial\tau^a}$ in the order $\mathcal{O}(G^{-1})$. The first step in the construction of perturbation theory is a transformation, raising the order of these operators:

$$\Phi[u,u_n] \longrightarrow e^{iG^2 J(\tilde\tau)}\Phi[u,u_n] \Longrightarrow i\frac{\partial}{\partial\tau^a} \longrightarrow -G^2 J_a + i\frac{\partial}{\partial\tau^a}, \qquad \text{where} \quad J_a \equiv \frac{\partial J(\tilde\tau)}{\partial\tau^a}. \tag{15}$$

Now the operators $\hat{q}(\lambda')$ and $\hat{p}(\lambda')$, expressed through $v(\lambda')$ and $u(\lambda')$, are

$$\hat{q}(\lambda') = G\,F(\lambda') + \hat{Q}(\lambda') + \frac{1}{G}\hat{A}(\lambda') + \mathcal{O}(G^{-2}), \qquad \hat{p}(\lambda') = G\,F_n(\lambda') + \hat{P}(\lambda') + \frac{1}{G}\hat{A}_n(\lambda') + \mathcal{O}(G^{-2}),$$

where

$$F(\lambda') = \frac{1}{\sqrt{2}}\Big(v(\lambda') - J_a\tilde{N}^a(\lambda')\Big),$$

$$\hat{Q}(\lambda') = \frac{1}{\sqrt{2}}\left(u(\lambda') + i\frac{\delta}{\delta u_n(\lambda')} + \tilde{N}^a(\lambda')r_a\right), \qquad \hat{P}(\lambda') = \frac{1}{\sqrt{2}}\left(u_n(\lambda') - i\frac{\delta}{\delta u(\lambda')} + \tilde{N}^a_n(\lambda')r_a\right),$$

$$\hat{A}(\lambda') = \frac{1}{\sqrt{2}}\tilde{N}^b(\lambda')\left(i\hat{S}_b + R^a_b r_a + i\frac{\partial}{\partial\tau^b}\right), \qquad r_a \equiv R^c_a J_c.$$

3.2 Perturbation theory and integrals of motion

The next step in the construction of perturbation theory is finding integrals of motion. As ten independent integrals of motion we select the following ones ($\alpha < \beta$):

$$\mathcal{P}^\beta = \int \Theta^{0\beta}\,dx_1dx_2dx_3, \qquad \mathcal{M}^{\alpha\beta} = \int\Big(\Theta^{0\alpha}x^\beta - \Theta^{0\beta}x^\alpha\Big)\,dx_1dx_2dx_3, \quad \text{where} \quad \Theta^{\alpha\beta} = \frac{\partial\mathcal{L}}{\partial f_{,\alpha}}\frac{\partial f}{\partial x_\beta} - g^{\alpha\beta}\mathcal{L}.$$

We have to note, that the integrals $\mathcal{M}^{\alpha\beta}$ are not translation-invariant. If the origin of coordinates removes on τ^μ, then the integrals $\mathcal{M}^{\alpha\beta}$ change on $\tau^\beta\mathcal{P}^\alpha - \tau^\alpha\mathcal{P}^\beta$.

We restrict our consideration to the case when C is the hyperplane $t' = 0$ and a coordinate system on C is $\lambda'^k = x'^k$, $k = 1,2,3$; so the normal derivative $\frac{\partial}{\partial\mathbf{n}} = \frac{\partial}{\partial t'}$. In what follows we shall show that it is possible to determine the functions $\tilde{N}^a(\mathbf{x}')$, $M_a(\mathbf{x}')$ and $u(\mathbf{x}')$ so that conditions (7) and (8) will be independent of choice of the hyperplane C.

Make quantization, changing the functions $f(\mathbf{x}')$ and $f_{,\alpha}(\mathbf{x}')$ to the operators:

$$f(0,x') \longrightarrow \hat{q}(\lambda') = \hat{q}(\mathbf{x}(0,x',\tilde\tau)), \qquad f_{,k}(0,x') \longrightarrow \hat{q}_{,k}(\mathbf{x}(0,x',\tilde\tau)), \qquad f_{,0}(0,x') \longrightarrow \hat{p}(\mathbf{x}(0,x',\tilde\tau)).$$

Now integrals of motion are series in $\frac{1}{G}$:

$$O^a = G^2 O^a_{-2} + G O^a_{-1} + O^a_0 + \dots \qquad a = 1,2,\dots,10.$$

Write in explicit form the first two terms in expression of the Hamiltonian: $\mathcal{P}^0 \equiv \mathcal{H} = G^2\mathcal{H}_{-2} + G\mathcal{H}_{-1} + \dots$, where[6]

$$\begin{aligned}
\mathcal{H}_{-2} &= \int_C \Big(\tfrac{1}{2}(F_{,0}^2 + F_{,1}^2 + F_{,2}^2 + F_{,3}^2) + V(F)\Big)\,d^3S,\\
\mathcal{H}_{-1} &= \int_C \Big(\tfrac{1}{2}(F_{,0}\hat{P} + F_{,1}\hat{Q}_{,1} + F_{,2}\hat{Q}_{,2} + F_{,3}\hat{Q}_{,3}) + V'(F)\hat{Q}\Big)\,d^3S.
\end{aligned}$$

Since $F(\mathbf{x}',\tilde\tau)$ is a number function, not an operator, so the integrals O^a_{-2} are some numbers as well. The integrals O^a_{-1} are linear in the operators u, u_n, $\frac{\delta}{\delta u}$ and $\frac{\delta}{\delta u_n}$. These operators have no normalized

[6]We differentiate over x': $F_{,\alpha} \equiv \frac{\partial F}{\partial x'^\alpha}$, $Q_{,\alpha} \equiv \frac{\partial Q}{\partial x'^\alpha}$ and $A_{,\alpha} \equiv \frac{\partial A}{\partial x'^\alpha}$.

eigenvectors, so to construct regular perturbation theory we have to vanish all integrals O^a_{-1}. As in the case of the $(1+1)$–dimensional theory [8], the operators O^a_{-1} have to be linear combinations of $\omega(\tilde{N}^a, u)$ and left parts of formulas (10). It is possible if $F(\mathbf{x}', \tilde{\tau})$ is a solution of the following equation:

$$g^{\alpha\beta}\frac{\partial^2 F(\mathbf{x}',\tilde{\tau})}{\partial x'^\alpha \partial x'^\beta} + V'(F) = 0. \tag{2'}$$

On the hyperplane C the following relations have to be satisfied:

$$v(\lambda') = -\tilde{N}^a(\lambda')J_a = \frac{1}{\sqrt{2}}F(\lambda') \Longrightarrow F(\lambda') = -\sqrt{2}\tilde{N}^a(\lambda')J_a. \tag{16}$$

We also have to impose the boundary conditions $F_{,\alpha}\hat{Q}|_{\partial C} = 0,\ \ x'^\alpha F_{,\beta}\hat{Q}|_{\partial C} = 0.$

A consequence of (16) is $J_a = -\frac{1}{\sqrt{2}}\omega(F, M_a)$.

Up to this point we have considered functions only on the three-dimensional hyperplane C. Now we propagate our consideration on the Minkowski space. Let $v(\mathbf{x}', \tilde{\tau}) = \frac{1}{\sqrt{2}}F(\mathbf{x}', \tilde{\tau})$, then $v(\mathbf{x}', \tilde{\tau})$ is a function, which is proportional to a solution of equation $(2')$, therefore, $M_a(\mathbf{x}', \tilde{\tau})$ are solutions of the following linear equation:

$$g^{\alpha\beta}\frac{\partial^2 M_a(\mathbf{x}',\tilde{\tau})}{\partial x'^\alpha \partial x'^\beta} + V''(F)\cdot M_a(\mathbf{x}',\tilde{\tau}) \equiv \hat{L}M_a(\mathbf{x}',\tilde{\tau}) = 0. \tag{17}$$

Let $\tilde{C}$ be a space-like hyperplane and S be domain of the space-time, restricted by the hyperplanes C and $\tilde{C}$. That is to say, the boundary of S is $\partial S = C + \tilde{C} + \partial\tilde{S}$. Superpose on functions $\tilde{N}^a$ and M_b such boundary conditions as space coordinates $x' \to \infty$ that $\int\limits_{\partial\tilde{S}} (\tilde{N}^a M_{bn} - M_b \tilde{N}^a_n)\, d^3S = 0$.

If $\tilde{N}^a$ also are solutions of equation (17), then, using Green formula, we obtain

$$\oint\limits_{\partial S}(\tilde{N}^a M_{bn} - M_b\tilde{N}^a_n)\, d^3S = \int\limits_S (\tilde{N}^a\hat{L}M_b - M_b\hat{L}\tilde{N}^a)\, d^4S = 0.$$

Therefore, conditions (8) do not depend on choice of a space-like hyperplane. Conditions (7) are also independent of choice of this hyperplane if $u(\mathbf{x}', \tilde{\tau})$ is a solution of (17) with suitable boundary conditions. The choice of the hyperplane of integration determines only values of the parameters J_a.

3.3 Determination of the functions $\tilde{N}^a$

Up to this point the functions $\tilde{N}^a(\mathbf{x}')$ are not determined. Since these functions are solutions of equation (17), they are uniquely determined if values of $\tilde{N}^a(0, x')$ and $\tilde{N}^a_n(0, x')$ are specified.

Let the function $F(\mathbf{x}')$ be zero on some space-like hyperplane. Without the loss of generality we can assume that $F(\mathbf{x}')$ is zero on the hyperplane C, determined by the equation $t' = 0$. Since $F(0, x') = 0$, then $F_{,k}(0, x') = 0$ and $F_{,kk}(0, x') = 0\ \ (k = 1, 2, 3)$. From equation $(2')$ and the condition $V'(0) = 0$ it follows, that $F_{,00}(0, x') = 0$. We also assume, that $F(0, x')$ is either even or odd function.

Using values of $M_a(0, x')$ and $M_{an}(0, x')$ we specify values of $\tilde{N}^a(0, x')$ and $\tilde{N}^a_n(0, x')$:

$$\begin{array}{llll}
M_0 = \frac{1}{\sqrt{2}}F_{,0}; & M_{0n} = 0; & \tilde{N}^0 = 0; & \tilde{N}^0_n = B_0 F_{,0};\\
M_1 = 0; & M_{1n} = \frac{1}{\sqrt{2}}F_{,01}; & \tilde{N}^1 = -B_1 F_{,01}; & \tilde{N}^1_n = 0;\\
M_2 = 0; & M_{2n} = \frac{1}{\sqrt{2}}F_{,02}; & \tilde{N}^2 = -B_2 F_{,02}; & \tilde{N}^2_n = 0;\\
M_3 = 0; & M_{3n} = \frac{1}{\sqrt{2}}F_{,03}; & \tilde{N}^3 = -B_3 F_{,03}; & \tilde{N}^3_n = 0;\\
M_4 = \frac{1}{\sqrt{2}}x'^1 F_{,0}; & M_{4n} = 0; & \tilde{N}^4 = 0; & \tilde{N}^4_n = B_4\Big(x'^1 F_{,0} - D_1 F_{,01}\Big);\\
M_5 = \frac{1}{\sqrt{2}}x'^2 F_{,0}; & M_{5n} = 0; & \tilde{N}^5 = 0; & \tilde{N}^5_n = B_5\Big(x'^2 F_{,0} - D_2 F_{,02}\Big);\\
M_6 = \frac{1}{\sqrt{2}}x'^3 F_{,0}; & M_{6n} = 0; & N^6 = 0; & \tilde{N}^6_n = B_6\Big(x'^3 F_{,0} - D_3 F_{,03}\Big);\\
M_7 = 0; & M_{7n} = \frac{1}{\sqrt{2}}\Big(x'^1 F_{,02} - x'^2 F_{,01}\Big); & \tilde{N}^7 = B_7\Big(x'^2 F_{,01} - x'^1 F_{,02}\Big); & \tilde{N}^7_n = 0;\\
M_8 = 0; & M_{8n} = \frac{1}{\sqrt{2}}\Big(x'^3 F_{,01} - x'^1 F_{,03}\Big); & \tilde{N}^8 = B_8\Big(x'^1 F_{,03} - x'^3 F_{,01}\Big); & \tilde{N}^8_n = 0;\\
M_9 = 0; & M_{9n} = \frac{1}{\sqrt{2}}\Big(x'^2 F_{,03} - x'^3 F_{,02}\Big); & \tilde{N}^9 = B_9\Big(x'^3 F_{,02} - x'^2 F_{,03}\Big); & \tilde{N}^9_n = 0;
\end{array}$$

where $D_k = \dfrac{\int x'^k F_{,0} F_{,0k}\, d^3S}{\int F^2_{,0k}\, d^3S}$, $k = 1,2,3$. It is easy to verify, that if we select such constants B_a that[7] $\omega(M_a, \tilde{N}^a) = 1$, then conditions (8) are satisfied.

3.4 Reduction of field states

When we have done the Bogolyubov transformation, we have added new independent variables $\tilde{\tau}$. To compensate appearance of new degrees of freedom we have imposed the subsidiary conditions on $u(\mathbf{x}(\mathbf{x}', \tilde{\tau}))$. Beginning with expressions (11) we consider the functions $f(\mathbf{x}')$ and $f_n(\mathbf{x}')$ as independent, so we duplicate field states number. Hence, we duplicate number of independent variables, including the Bogolyubov variables $\tilde{\tau}$. So correct reduction of field states has to be made on quantum functions, satisfying not 10, but 20 subsidiary conditions. Ten additional conditions can be chosen in the following way: let's consider such functions $w(\mathbf{x}', \tilde{\tau})$ that

$$\forall a = 0..9\ :\ \omega(M_a, w) = 0, \qquad \omega(\tilde{N}^a, w) = 0. \tag{7'}$$

The function w can be obtained from an arbitrary differentiable function y, using the following formula:

$$w = y + \tilde{N}^a \cdot \omega(M_a, y) - M_a \cdot \omega(\tilde{N}^a, y) - \tilde{N}^a \cdot \omega(M_a, M_b)\omega(\tilde{N}^b, y), \tag{9'.1}$$

hence,

$$w_n = y_n + \tilde{N}^a_n \cdot \omega(M_a, y) - M_{an} \cdot \omega(\tilde{N}^a, y) - \tilde{N}^a_n \cdot \omega(M_a, M_b)\omega(\tilde{N}^b, y). \tag{9'.2}$$

There exist such constants $\mathcal{C}^a$, C^a_b, $\tilde{\mathcal{C}}^a$ and $\tilde{C}^a_b$ that[8]:

$$\tilde{N}^a(0, x') = \mathcal{C}^a M_{an}(0, x') + C^a_b \tilde{N}^b_n(0, x'), \qquad \tilde{N}^a_n(0, x') = \tilde{\mathcal{C}}^a M_a(0, x') + \tilde{C}^a_b \tilde{N}^b(0, x'). \tag{18}$$

Therefore, the functions $w(0, x')$ and $w_n(0, x')$ are restricted by the same subsidiary conditions and are results of action by the same projecting operator on function y and y_n respectively.

The derivatives over $w(\lambda')$ and $w_n(\lambda')$ satisfy the following conditions:

$$\int\limits_C \left(M_a(\lambda') \frac{\delta}{\delta w(\lambda')} + M_{an}(\lambda') \frac{\delta}{\delta w_n(\lambda')} \right) dS = 0, \qquad \int\limits_C \left(\tilde{N}^a(\lambda') \frac{\delta}{\delta w(\lambda')} + \tilde{N}^a_n(\lambda') \frac{\delta}{\delta w_n(\lambda')} \right) dS = 0.$$

To compensate new subsidiary conditions, we treat r_a as independent variables. The variables r_a have no physical sense. They have appeared as a remainder of the state space reduction in terms of Bogolyubov group variables. Since

$$r_a \equiv R^b_a J_b = \omega(\tilde{N}^b_a, u) J_b = -\omega(M_a, u),$$

then functions, satisfying conditions (7′), are

$$w(\lambda') = u(\lambda') - \tilde{N}^a(\lambda') r_a, \qquad w_n(\lambda') = u_n(\lambda') - \tilde{N}^a_n(\lambda') r_a.$$

In other points of the Minkowski space the function w is determined as a solution of (17).

The various derivatives $\frac{\delta}{u(\lambda')}$ and $\frac{\delta}{u_n(\lambda')}$ can be expressed through derivatives over new variables:

$$\frac{\delta}{\delta u(\lambda')} = \frac{\delta}{\delta w(\lambda')} + \frac{\delta r_a}{\delta u(\lambda')} \frac{\partial}{\partial r_a}, \qquad \frac{\delta}{\delta u_n(\lambda')} = \frac{\delta}{\delta w_n(\lambda')} + \frac{\delta r_a}{\delta u_n(\lambda')} \frac{\partial}{\partial r_a},$$

where

$$\frac{\delta r_a}{\delta u(\lambda')} = -M_{an}(\lambda') + \omega(M_a, M_b) \tilde{N}^b_n(\lambda'), \qquad \frac{\delta r_a}{\delta u_n(\lambda')} = M_a(\lambda') - \omega(M_a, M_b) \tilde{N}^b(\lambda').$$

The reduction of state number can be realized due to the choice of state vectors in the form $\Phi[w, w_n] = exp\Big(-i \int w_n w\, d^3S\Big) \breve{\Phi}[w]$, now the operators $\hat{P}$ and $\hat{Q}$ are

$$\hat{P}(\lambda') = -\frac{i}{\sqrt{2}} \left(\frac{\delta}{\delta w(\lambda')} + \frac{\delta r_a}{\delta u(\lambda')} \frac{\partial}{\partial r_a} \right), \qquad \hat{Q}(\lambda') = \sqrt{2} \left(w(\lambda') + \frac{i}{2} \frac{\delta r_a}{\delta u_n(\lambda')} \frac{\partial}{\partial r_a} \right). \tag{20}$$

[7] We do not summarize on index a.

[8] In formulas (18) we summarize only on index b.

3.5 The zeroth order in G

The integrals of motion in the zeroth order are

$$\mathcal{H}_0 = i\frac{\partial}{\partial\tau^0} + \mathcal{H}_{01} + \mathcal{H}_{02} + \mathcal{H}_{03}, \qquad \mathcal{P}_0^k = -i\frac{\partial}{\partial\tau^k} + \mathcal{P}_{01}^k + \mathcal{P}_{02}^k + \mathcal{P}_{03}^k,$$

$$\mathcal{M}_0^{0k} = i\frac{\partial}{\partial\tau^{3+k}} + \tau^0\mathcal{P}_0^k - \tau^k\mathcal{H}_0 + \mathcal{M}_{01}^{0k} + \mathcal{M}_{02}^{0k} + \mathcal{M}_{03}^{0k},$$

$$\mathcal{M}_0^{kj} = i\frac{\partial}{\partial\tau^{4+k+j}} - \tau^k\mathcal{P}_0^j + \tau^j\mathcal{P}_0^k + \mathcal{M}_{01}^{kj} + \mathcal{M}_{02}^{kj} + \mathcal{M}_{03}^{kj}.$$

Operators O_{01}^a, O_{02}^a and O_{03}^a act on space of functionals $\Phi[w, w_n]\tilde{\Phi}[r]$, with operators O_{01}^a and O_{02}^a act in the space of $\Phi[w, w_n]$, whereas operators O_{03}^a act in the orthogonal space of $\tilde{\Phi}[r]$. It is easy to show, that for all a sums of the integrals O_{01}^a and O_{02}^a are zero.

The operators O_{03}^a include only the superfluous variables r_a and $\frac{\partial}{\partial r_a}$, they can be removed due to corresponding choice of state vectors [8].

After removing the superfluous variables we obtain

$$\mathcal{H}_0 = i\frac{\partial}{\partial\tau^0}, \qquad \mathcal{P}_0^k = -i\frac{\partial}{\partial\tau^k},$$

$$\mathcal{M}_0^{0k} = i\left(\frac{\partial}{\partial\tau^{3+k}} - \tau^0\frac{\partial}{\partial\tau^k} - \tau^k\frac{\partial}{\partial\tau^0}\right), \qquad \mathcal{M}_0^{kj} = i\left(\frac{\partial}{\partial\tau^{4+k+j}} + \tau^k\frac{\partial}{\partial\tau^j} - \tau^j\frac{\partial}{\partial\tau^k}\right).$$

Hence, the integrals of motion in the zeroth-order approximation satisfy the commutation relations for generators of the Poincaré group and generate the Lie algebra of this group.

The field operator is

$$\phi(\mathbf{x}') = GF(\mathbf{x}(\mathbf{x}', \tilde{\tau})) + \hat{\psi}(\mathbf{x}(\mathbf{x}', \tilde{\tau})) + \frac{\delta r_a}{\delta u_n}\frac{\partial}{\partial r_a} + \frac{1}{G}\hat{A}(\mathbf{x}(\mathbf{x}', \tilde{\tau})) + \mathcal{O}(G^{-2}), \tag{21}$$

where $\mathbf{x}$ and $\mathbf{x}'$ are connected by (3), $\hat{\psi}(\mathbf{x}', \tilde{\tau})$ is a solution of the following Cauchy problem:

$$g^{\alpha\beta}\frac{\partial^2\hat{\psi}(\mathbf{x}', \tilde{\tau})}{\partial x'^\alpha \partial x'^\beta} + V''(F)\hat{\psi}(\mathbf{x}', \tilde{\tau}) = 0, \qquad \hat{\psi}(\mathbf{x}', \tilde{\tau})|_C = \sqrt{2}w(\lambda'), \qquad \hat{\psi}_{t'}(\mathbf{x}', \tilde{\tau})|_C = \sqrt{2}w_{t'}(\lambda').$$

Conclusion

In the present report quantization close to nonstationary classical field has been done. We have shown that the transformation, introduced by N.N. Bogolyubov, allows one to use variables having the meaning of symmetry group parameters. Using Bogolyubov group variables we have constructed perturbation theory, which does not violate the conservations laws. The expressions of the integrals of motions and the field operator ave been obtained with a precision of zeroth order with respect to inverse powers of the coupling constant G. So, completely Poincaré-invariant description of the $(3+1)$–dimensional quantum system with nontrivial classical component has been given.

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