Howe dual pairs and the structure of gauge orbit spaces

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Abstract

We apply the notion of Howe dual pairs and Howe subgroups to the classification of orbit types of the gauge group action in the space of gauge potentials for $\mathrm{SU}(n)$ gauge theories on compact Riemannian manifolds M of dimension $2 \leq d \leq 4$. This allows us to relate the orbit types of this action to subbundles of a principal $\mathrm{SU}(n)$-bundle P, the structure groups of which are Howe subgroups of $\mathrm{SU}(n)$. We construct characteristic classes classifying these subbundles and give an explicit description of the orbit types in terms of elements of the cohomology groups of M.

1 Introduction

The physical configuration space of non-abelian gauge theories, which is the space of gauge group orbits in the space of gauge potentials, may have a highly nontrivial structure. The crucial point is that, in general, the orbit space contains not only the orbits of principal orbit type, but also orbits of other types, which represent singularities of the configuration space. These singularities affect the classical motion on the orbit space due to boundary conditions and, in this way, may also affect the quantum properties of non-abelian gauge theories [1-4].

So far, the orbit types have been determined only for SU(2)-gauge theory over space-time manifolds S^2 [5] and S^4 [6]. Another approach to studying the singularity structure of the gauge orbit space is to use gauge conditions and some additional techniques to find a fundamental modular domain and then to analyze the boundary of this domain [3, 4].

A more general approach, which is applicable to arbitrary gauge groups and space-time manifolds and relates the orbit types of the gauge group action to certain bundle reductions, was proposed in [5, 7]. In the present contribution we are going to make a new step in this direction and to give a complete description of the stratification structure of the gauge orbit space for $\mathrm{SU}(n)$ gauge theories in Euclidean space-time of dimension $d = 2, 3, 4$ in terms of cohomology groups of the space-time manifold.

2 Orbit Types and Bundle Reductions

We consider a gauge theory with gauge group G on a Riemannian manifold M and restrict ourselves to a certain topological sector, i.e. to a certain principal G-bundle P over M. We denote the set of smooth connection forms on P by $\mathcal{C}$, and the group of gauge transformations, viewed as equivariant mappings $g : P \to G, \quad g(pa) = a^{-1}g(p)a$, by $\mathcal{E}$. The right action of $g \in \mathcal{E}$ on $\omega \in \mathcal{C}$ is given by

$$\omega^{(g)} = \mathrm{Ad}(g^{-1})\omega + g^*\theta_G \,. \tag{1}$$

Here θ_G is the Maurer-Cartan form of G and g^* denotes the pull-back. The set $\mathcal{C}$ can be completed to an affine Hilbert space, and $\mathcal{E}$ can be completed to an infinite dimensional Lie group in such a way that the action (1) is smooth [7, 8]. The orbit space $\mathcal{C}/\mathcal{E}$ will be denoted by $\mathcal{M}$.

The stabilizer (or isotropy subgroup) of a connection $\omega \in \mathcal{C}$ is the subgroup $\mathcal{E}_\omega = \{g \in \mathcal{E} \,|\, \omega^{(g)} = \omega\}$. Since the elements of $\mathcal{E}_\omega$ map ω-horizontal paths in P to ω-horizontal paths, they are constant on the holonomy bundle $P(\omega, p)$ of ω based at some $p \in P$ and coincide there with the structure group transformations. Conversely, any gauge transformation, which is constant on $P(\omega,p)$, leaves ω invariant. Thus,

$$\mathcal{E}_\omega = \{g \in \mathcal{E} |\, g|_{P(\omega,p)} = \text{const}\}\,, \tag{2}$$

and the elements of $\mathcal{E}_\omega$ are uniquely determined by their value at an arbitrary point $p \in P$. Moreover, the elements of $\mathcal{E}_\omega$ being equivariant mappings, their restrictions to $P(\omega,p)$ must take values in the centralizer in G of the holonomy group $H(\omega,p)$ of ω based at p. Vice versa, an element of this centralizer defines a constant function on $P(\omega,p)$ which can be extended equivariantly to a gauge transformation on P. Thus, the map

$$\mathcal{E}_\omega \to C_G(H(\omega,p))\,, g \mapsto g(p) \tag{3}$$

is a group isomorphism.

Since the stabilizer of $\omega \in \mathcal{C}$ and that of a gauge transform $\omega^{(g)}$ are connected by $\mathcal{E}_{\omega^{(g)}} = g^{-1}\mathcal{E}_\omega g$, there is a natural map assigning to each orbit in $\mathcal{C}$ (i. e. to each point of $\mathcal{M}$) the conjugacy class in $\mathcal{E}$ of the stabilizers of its representatives. The image is called the type of this orbit. Let $\mathrm{OT}(\mathcal{C},\mathcal{E})$ denote the set of orbit types of the $\mathcal{E}$-action on $\mathcal{C}$. This set carries a natural partial ordering: $\sigma \geq \sigma'$ iff there are representatives $\mathcal{E}_\omega$ of σ and $\mathcal{E}_{\omega'}$ of σ' such that $\mathcal{E}_\omega \subseteq \mathcal{E}_{\omega'}$. With respect to this partial ordering, there is exactly one maximal element, namely, the class ζ, whose stabilizer is the subgroup of $\mathcal{E}$ made up by the functions which are constant on the whole of P (and hence necessarily take values in the center $Z(G)$). The subsets $\mathcal{M}_\tau \subseteq \mathcal{M}$ consisting of gauge orbits of type τ can be equipped with a smooth manifold structure [7], and the decomposition $\mathcal{M} = \bigcup_{\tau\in\mathrm{OT}(\mathcal{C},\mathcal{E})} \mathcal{M}_\tau$ is a stratification, the manifolds $\mathcal{M}_\tau$ being called strata. The stratum $\mathcal{M}_\tau$ is open and dense in $\bigcup_{\tau'\leq\tau} \mathcal{M}_{\tau'}$, and, in this sense, the strata $\mathcal{M}_{\tau'}$, $\tau' < \tau$, are singularities in the union. Since there exists the principal orbit type $\zeta \geq \tau, \forall \tau \in \mathrm{OT}(\mathcal{C},\mathcal{E})$, all the strata of non-principal orbit type constitute singularities in the total orbit space $\mathcal{M}$. As we have already mentioned in the Introduction, these singularities can affect both classical and quantum properties of gauge theories. Therefore, in order to study this in more detail, it is necessary, in a first step, to find out, what singularities are present and how they are located with respect to each other. According to our reasoning, this amounts to the determination of the partially ordered set $\mathrm{OT}(\mathcal{C},\mathcal{E})$.

We have seen that stabilizers of $\mathcal{E}$-action can be characterized by the subgroups of gauge transformations which are constant on holonomy subbundles (2). This suggests the idea to characterize orbit types by certain classes of subbundles of P [5]. To realize this idea, let us first introduce a new definition.

An ordered pair (H_1, H_2) of subgroups of G, which obeys $C_G(H_1) = H_2$, $C_G(H_2) = H_1$, is called a *Howe dual pair* [9]. The subgroups H_1 and H_2 are called *Howe subgroups*. Equivalently, a Howe subgroup can be defined by the relation $C^2_G(H) = H$, where $C^2_G(H)$ denotes the double centralizer. Accordingly, we will call a subbundle of P, the structure group of which is Howe, a *Howe subbundle*.

Now let Q be a Howe subbundle of P with structure group H. This subbundle is called holonomy-induced iff it contains a connected subbundle $\tilde{Q} \subset Q$ with structure group $\tilde{H}$, which obeys $C^2_G(\tilde{H}) = H$.

If Q is holonomy-induced then the subgroup $\mathcal{E}(Q) = \{g \in \mathcal{E} |\, g|_Q = \mathit{const}\}$ is a stabilizer. To see this, recall that on $\tilde{Q}$, as on any connected principal bundle over a paracompact space, there exists a connection $\tilde{\omega}$ with full holonomy [10, Ch. II, Thm. 8.2]. By equivariant extension, $\tilde{\omega}$ defines a connection ω on P with holonomy bundle $\tilde{Q}$. According to the definition of the holonomy-induced Howe subbundles, $\mathcal{E}(Q) = \mathcal{E}(\tilde{Q}) = \mathcal{E}_\omega$. This equality and isomorphism (3) show that the set of holonomy-induced Howe subbundles is in one-to-one correspondence with the set of the stabilizers of $\mathcal{E}$-action. Now in order to describe the set $OT(\mathcal{C},\mathcal{E})$ in terms of Howe subbundles, we have to introduce an equivalence relation on this set, which corresponds to passing to conjugacy classes of the stabilizers of $\mathcal{E}$-action. This can be easily done due to the following transformation property

$$\mathcal{E}\left(Q^{(g)} \cdot a\right) = g\mathcal{E}(Q)g^{-1}\,, \qquad \forall\, g \in \mathcal{E}, a \in G\,. \tag{4}$$

Thus, to describe the set $\mathrm{OT}(\mathcal{C},\mathcal{E})$ in terms of Howe subbundles, we have to classify the holonomy-induced Howe subbundles of P up to isomorphy, i.e. to factor out the actions of the gauge group $\mathcal{E}$ and the structure group G.

Let Howe(P) denote the set of isomorphism classes of holonomy-induced Howe subbundles of P. This set carries a natural partial ordering, which is defined by inclusion modulo conjugacy w.r.t. the action of G and $\mathcal{E}$. It is obvious that this partial ordering corresponds to that on $\mathrm{OT}(\mathcal{C},\mathcal{E})$.

Next we will describe the set Howe(P) for $G = \mathrm{SU}(n)$ and $\dim M = 2,3,4$. To this end we have to classify Howe subgroups of $\mathrm{SU}(n)$ first.

3 The Howe Subgroups of $\mathrm{SU}(n)$

Let $\mathrm{K}(n)$ denote the collection of pairs of r-vectors, $r = 1,2,\ldots,n$, with positive integer components

$$J = (\mathbf{k},\mathbf{l}) = ((k_1,\ldots,k_r),(l_1,\ldots,l_r)), \quad \mathbf{kl} = \sum_{i=1}^{r} k_i l_i = n\,.$$

Any $J \in \mathrm{K}(n)$ defines an embedding $\mathrm{U}(k_1)\times\cdots\times\mathrm{U}(k_r) \hookrightarrow \mathrm{U}(n)$, which can be explicitly represented as follows:

$$(A_1,\ldots,A_r) \mapsto \begin{pmatrix} \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & A_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_1 \end{pmatrix} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \begin{pmatrix} A_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & A_r \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_r \end{pmatrix} \end{pmatrix} \tag{5}$$

(l_1 times A_1, l_2 times A_2, etc.). The image is denoted by $\mathrm{U}(J)$. It can be shown that the centralizer of $\mathrm{U}(J)$ in $\mathrm{U}(n)$ is conjugate to $\mathrm{U}(J')$, where $J' = (\mathbf{l},\mathbf{k})$ [11]. Hence $\mathrm{U}(J)$ is Howe in $\mathrm{U}(n)$. Then the intersection $\mathrm{SU}(J) = \mathrm{U}(J)\cap\mathrm{SU}(n)$ is Howe in $\mathrm{SU}(n)$. Direct evaluation of the determinant of the matrix (5) leads to the result that the group $\mathrm{SU}(J)$ has g connected components, g denoting the greatest common divisor of $l_1,\ldots,l_r$. Moreover, it is easy to see that if J' is obtained from $J = (\mathbf{k},\mathbf{l})$ by a permutation of vector components, then $\mathrm{SU}(J)$ and $\mathrm{SU}(J')$ are conjugate in $\mathrm{SU}(n)$. We denote the set $\mathrm{K}(n)$, modulo permutations of the vector components, by $\hat{\mathrm{K}}(n)$ and its elements by $\hat{J} = [J]$. Thus, the assignment $\hat{J} \mapsto [\mathrm{SU}(J)]$ is a bijection of the set $\hat{\mathrm{K}}(n)$ onto the set of the conjugacy classes of Howe subgroups of $\mathrm{SU}(n)$. Due to this property, the set $\hat{\mathrm{K}}(n)$ carries a partial ordering, which is induced by the natural partial ordering on the set of Howe subgroups, the latter being inclusion modulo conjugation.

It is obvious that the classifying set $\hat{\mathrm{K}}(n)$ has unique minimal and maximal elements with respect to this partial ordering. Consider the following element of $\mathrm{K}(n)$: $J_{\min} = (1,n)$. It is easy to understand that $\mathrm{U}(J_{\min})$ can be embedded in any $\mathrm{U}(J)$ of the form (5), and hence $\mathrm{SU}(J_{\min})$ can be embedded in any $\mathrm{SU}(J)$. Therefore, $[J_{\min}] \leq [J]$, and $[J_{\min}]$ is the unique minimal element of $\hat{\mathrm{K}}(n)$. It is also clear that $[J_{\min}]$ corresponds to the center of $\mathrm{SU}(n)$.

Next, let us consider $J_{\max} = C(J_{\min}) = (n,1)$. Since $\mathrm{SU}(J_{\max}) = \mathrm{SU}(n)$, it is the unique maximal element of $\hat{\mathrm{K}}(n)$. The set $\hat{\mathrm{K}}(n)$ being finite, one can derive its partial ordering by the determination of direct successors starting from the minimal element. Therefore, next we are going to derive the direct successors of the elements of $\hat{\mathrm{K}}(n)$, which are characterized by the following two operations:

Splitting Let $J \in \mathrm{K}(n)$, choose i between 1 and $r < n$ such that $l_i \neq 1$ and a decomposition $l_i = l_{i_1} + l_{i_2}$ with positive integers l_{i_1}, l_{i_2}. Define $J' = (\mathbf{k}',\mathbf{l}')$ by a pair of $(r+1)$-vectors

$$\mathbf{k}' = (k_1,\ldots,k_{i-1},k_i,k_i,k_{i+1},\ldots,k_r)\,,\, \mathbf{l}' = (l_1,\ldots,l_{i-1},l_{i1},l_{i2},l_{i+1},\ldots,l_r)\,,$$

i.e. by replacing the component l_i of $\mathbf{l}$ by the two components l_{i_1}, l_{i_2} and doubling the component k_i of $\mathbf{k}$. By construction, $J' \in \mathrm{K}(n)$. We shall say that J' arises from J by splitting l_i into l_{i_1} and l_{i_2}.

Obviously, $J \leq J'$ and on the level of Howe subgroups the corresponding embedding is realized by the homomorphism, which maps the factor $\mathrm{U}(k_i)$ of a matrix of the form (5) diagonally in $\mathrm{U}(k_i) \times \mathrm{U}(k_i)$ of another such matrix and is equal to identity on all the other factors.

Merging Choose $J \in \mathrm{K}(n)$ with $r > 1$ such that $l_{i_1} = l_{i_2}$ for $i_1 < i_2$ between 1 and r. Define $J' = (\mathbf{k}', \mathbf{l}')$ by a pair of $(r-1)$-vectors

$$
\begin{aligned}
\mathbf{k}' &= \left(k_1, \ldots, k_{i_1-1}, k_{i_1} + k_{i_2}, k_{i_1+1}, \ldots, \widehat{k}_{i_2}, \ldots, k_r\right), \\
\mathbf{l}' &= \left(l_1, \ldots, l_{i_1-1}, l_{i_1}, l_{i_1+1}, \ldots, \widehat{l}_{i_2}, \ldots, l_r\right),
\end{aligned}
$$

i.e. by replacing the component k_{i_1} of $\mathbf{k}$ by $k_{i_1} + k_{i_2}$ and deleting the components k_{i_2} of $\mathbf{k}$ and l_{i_2} of J. As a consequence of $l_{i_2} = l_{i_1}$, we have $J' \in \mathrm{K}(n)$. Again we have $J \leq J'$, and the corresponding embedding is realized by the homomorphism, which maps the factor $\mathrm{U}(k_{i_1}) \times \mathrm{U}(k_{i_2})$ canonically into $\mathrm{U}(k_{i_1} + k_{i_2})$ and is equal to identity on all the other factors. We shall say that J' arises from J by merging k_{i_1} and k_{i_2}. One can check that the operations of splitting and merging are dual with respect to taking the centralizer, i.e. if J' arises from J by merging then $C(J)$ arises from $C(J')$ by splitting and vice versa. Now to find all the direct successors of J, we have to apply to it all possible splittings and mergings. Thus, we have described the Howe subgroups of $\mathrm{SU}(n)$ and formulated the rules for finding their partial ordering.

It is convenient to represent the conjugacy classes of Howe subgroups graphically by *Hasse diagrams.* Below we present the Hasse diagram of SU(3). The vertices are labeled by Howe subgroups of SU(3) and the corresponding elements of $\hat{\mathrm{K}}(3)$, and a line means that the left endpoint is less than the right endpoint. The subgroups, which lie symmetrically with respect to the middle point, are centralizers of each other, i.e form a Howe dual pair.

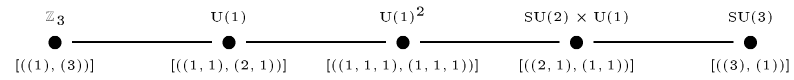


Figure 1: Hasse diagram of SU(3).

4 Howe Subbundles of $\mathrm{SU}(n)$-Bundles

Let P be a principal $\mathrm{SU}(n)$-bundle. First we note that all Howe subbundles of $\mathrm{SU}(n)$-bundles are holonomy-induced. Since the action of the structure group $\mathrm{SU}(n)$ on subbundles of P conjugates their structure groups, it suffices to consider subbundles Q with structure groups $\mathrm{SU}(J)$, $J \in \mathrm{K}(n)$; since two such subbundles are conjugate under the action of $\mathcal{E}$ if and only if they are isomorphic, it suffices to classify $\mathrm{SU}(J)$-subbundles of P up to isomorphy.

We consider the following homomorphisms:

$$
\begin{aligned}
j_J &: \mathrm{SU}(J) \to \mathrm{U}(J) && (\textit{natural embedding}), \\
\lambda_J &: \mathrm{SU}(J) \to \mathrm{SU}(J)/(\mathrm{SU}(J))_0 && (\textit{natural projection}).
\end{aligned}
$$

Here $(\mathrm{SU}(J))_0$ stands for the identity connected component of $\mathrm{SU}(J)$. We have already mentioned that $\mathrm{SU}(J)$ has g connected components, and therefore $\mathrm{SU}(J)/(\mathrm{SU}(J))_0 \cong \mathbb{Z}_g$. The homomorphisms j_J and λ_J give rise to associated bundles $Q \times_{\mathrm{SU}(J)} \mathrm{U}(J)$ and $Q/(\mathrm{SU}(J))_0$. Let us denote the set of isomorphism classes of G-bundles over M by $\mathrm{Bun}(M, G)$ and that of G-bundles reducible to $H \subseteq G$ by $\mathrm{Bun}(M, G_{\downarrow H})$. Under the assumption that the first homotopy group of M is torsion-free, the induced map

$$
\mathrm{Bun}(M, \mathrm{SU}(J)) \to \mathrm{Bun}(M, \mathrm{U}(J)_{\downarrow \mathrm{SU}(J)}) \times \mathrm{Bun}(M, \mathbb{Z}_g)
$$

is a bijection.

Recalling that $\mathrm{Bun}(M, \mathrm{U}(J)) = \prod_{i=1}^{r} \mathrm{Bun}(M, \mathrm{U}(k_i))$, we conclude that $\mathrm{Bun}(M, \mathrm{SU}(J))$ for $2 \leq \dim M \leq 4$ can be exactly described by invariants (α, ξ), where $\alpha = (\alpha_1, \ldots, \alpha_r)$, with α_i being total Chern classes of $\mathrm{U}(k_i)$-bundles over M, and $\xi \in H^1(M, \mathbb{Z}_g)$ classifies the $\mathbb{Z}_g$-bundles over M [12].

Let $Q \in \mathrm{Bun}(M, \mathrm{SU}(J))$ be given by invariants (α, ξ). Then Q is a subbundle of P iff

$$\alpha_1^{l_1} \smile \cdots \smile \alpha_r^{l_r} = c(P)\,, \tag{6}$$

where $\smile$ denotes the cup product, the powers of α_i are also taken with this product, and $c(P)$ denotes the total Chern class of P. We would like to note here that due to the facts that (i) ξ does not appear in this relation and (ii) the first homotopy group of M is torsion-free the cohomology classes can be represented by differential forms, and then the cup product will be represented by the exterior product.

In degree 2, formula (6) contains the condition for the $\mathrm{U}(J)$-bundle, characterized by α, to be reducible to $\mathrm{SU}(J)$.

Thus, a Howe subbundle Q of P is characterized by a triple $(J; \alpha, \xi)$, $\quad J \in \mathrm{K}(n)$, satisfying equation (6). As we have mentioned in the beginning of this section, the action of the structure group $\mathrm{SU}(n)$ on subbundles conjugates their structure groups, and therefore the Howe subbundles given by invariants $(J; \alpha, \xi)$ and $(\sigma J; \sigma\alpha, \xi)$, where σ is a permutation of $1, \ldots, r$, have to be identified. This concludes the classification of Howe subbundles of $\mathrm{SU}(n)$-bundles, and next we will consider an example of the group $\mathrm{SU}(3)$.

5 Example: SU(3)

Let P be a principal SU(3)-bundle. To find the Howe subbundles of P, we have to specify equations (6) for the case of the Howe subgroups of SU(3) and to solve them for particular choices of the base manifold M. Below we will consider the cases $M = S^4$ and $M = S^1 \times S^3$, which correspond to one-point compactifications of the Euclidean space R^4 and of the closed Robertson-Walker space-time $R \times S^3$.

The set $\hat{\mathrm{K}}(3)$ and the corresponding Howe subgroups are shown in Fig. 1. The $\mathrm{SU}(2) \times \mathrm{U}(1)$-subbundles are characterized by $\alpha_1 = 1 + \alpha_1^{(2)} + \alpha_1^{(4)}$ and $\alpha_2 = 1 + \alpha_2^{(2)}$, where $\alpha_i^{(2)} \in H^2(M, \mathbb{Z})$, $i = 1, 2$ and $\alpha_1^{(4)} \in H^4(M, \mathbb{Z})$. Equation (6) now reads $\alpha_1 \smile \alpha_2 = c(P)$. In degree 2, this yields $\alpha_2^{(2)} = -\alpha_1^{(2)}$. Then, in degree 4, we have

$$\alpha_1^{(4)} - \alpha_1^{(2)} \smile \alpha_1^{(2)} = c_2(P)\,. \tag{7}$$

The $\mathrm{U}(1) \times \mathrm{U}(1)$-subbundles are characterized by $\alpha_i = 1 + \alpha_i^{(2)}$, $i = 1, 2, 3$. Equation (6) reduces to $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and

$$\alpha_1^{(2)} \smile \alpha_2^{(2)} + \alpha_1^{(2)} \smile \alpha_3^{(2)} + \alpha_2^{(2)} \smile \alpha_3^{(2)} = c_2(P)\,. \tag{8}$$

Since the element $J = ((1,1,1),(1,1,1))$ is invariant under the permutations of vector components, the solutions, which differ in a permutation of the indices 1,2,3, have to be identified.

The U(1)-subbundles are characterized by $\alpha_i = 1 + \alpha_i^{(2)}$, $i = 1, 2$. In this case, equation (6) leads to $\alpha_1 + \alpha_2 = 0$ and

$$-\alpha_1^{(2)} \smile \alpha_1^{(2)} = c_2(P)\,. \tag{9}$$

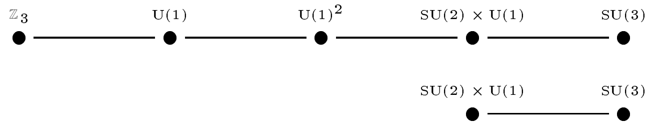
Finally, $\mathbb{Z}_3$-subbundles are given by $\xi \in H^1(M, \mathbb{Z}_3)$ and $\alpha = 1 + \alpha^{(2)}$. Equation (6) can be satisfied only in the case $c_2(P) = 0$ and then requires $\alpha^{(2)} = 0$.

Now let us consider these equations in the case $M = S^4$. Since both $H^2(M, \mathbb{Z})$ and $H^1(M, \mathbb{Z}_3)$ are trivial, $\xi = 0$ and all the $\alpha_i^{(2)} = 0$ as well. It is very easy to arrive at the following result. If $c_2(P) = 0$, each Howe subgroup of $\mathrm{SU}(n)$ gives rise to a unique Howe subbundle of P. If $c_2(P) \neq 0$, the only subbundle of P is that with the structure group $\mathrm{SU}(2) \times \mathrm{U}(1)$.

Howe subbundles of P can also be represented graphically by Hasse diagrams. Now the vertices correspond to the elements of $\mathrm{Howe}(P)$ and are labeled by their structure groups; the lines still mean that the left endpoint is less than the right endpoint.

$c_2(P) = 0$:

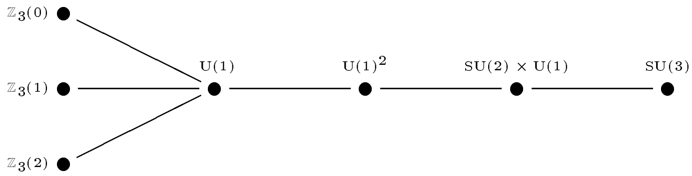
$c_2(P) \neq 0$:



It is obvious that each diagram can be interpreted as a stratification of the corresponding gauge orbit space. The vertex SU(3) represents the generic stratum, while all the other vertices correspond to the singularities.

In the case $M = S^1 \times S^3$ the second cohomology group $H^2(M, \mathbb{Z})$ is trivial, whereas $H^1(M, \mathbb{Z}_3) = \mathbb{Z}_3$. The only difference from the case of S^4 is that, for $c_2(P) = 0$, there are now three inequivalent $\mathbb{Z}_3$-subbundles. This is represented graphically by the following Hasse diagram:

$c_2(P) = 0$:



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