Inverse Scattering Approximate Method in Discrete Approach

S. A. Zaitsev
Khabarovsk State Technical University, Tikhookeanskaya 136, Khabarovsk 680035, Russia

Abstract

A method is suggested in the framework of $J-$matrix approach for construction of separable potentials with oscillator form factors from the $S-$matrix. As an example of the method, we invert $S-$matrix of np elastic scattering.

In this work, we construct a non-local separable potential

$$\widehat{V} = \hbar\omega \sum_{n,\, n'=0}^{N-1} \left|\varphi_n^l\right\rangle V_{n,\, n'} \left\langle\varphi_{n'}^l\right| \tag{1}$$

from the $s-$function. The potential is defined by an $N \times N$ matrix with elements $V_{i,\, j}$ and the parameter $\hbar\omega$. As form-factors are used the eigenfunction of the harmonic oscillator,

$$\left|\varphi_n^l\right\rangle = (-1)^n \sqrt{\frac{2n!}{\rho\, \Gamma(n+l+\frac{3}{2})}}\, \frac{x^{l+1}}{\exp(x^2/2)} L_n^{l+\frac{1}{2}}(x^2). \tag{2}$$

Here, x is the relative coordinate in units of the oscillation radius $\rho = \sqrt{\frac{\hbar}{\mu\omega}}$, and μ is the reduced mass. We measure the energy E in units of $\hbar\omega$, i.e., $E = \hbar\omega\epsilon$ and $\epsilon = q^2/2$, where q is the dimensionless momentum canonically conjugate to the coordinate x.

To solve the inverse-scattering problem we propose to use the so-called $J-$matrix approach (see for example [1]) in the framework of which continuum and bound state wave functions are expanded in an infinite series of oscillator functions (2). The expression for $s-$function corresponding to the potential (1) is of the form

$$s_l(E) = \frac{C^{(-)}_{N-1,l}(q) - \mathcal{P}_N(\epsilon) T^l_{N-1,N} C^{(-)}_{N,l}(q)}{C^{(+)}_{N-1,l}(q) - \mathcal{P}_N(\epsilon) T^l_{N-1,N} C^{(+)}_{N,l}(q)}, \tag{3}$$

where $C^{(\pm)}_{n,l}(q) = C_{n,l}(q) \pm \mathrm{i} S_{n,l}(q)$ are the discrete analogs of free waves [1]. Function $\mathcal{P}_N(\epsilon)$ is defined by the expression

$$\mathcal{P}_N(\epsilon) = \left[(\mathbf{I}\epsilon - \mathbf{h})^{-1} \right]_{N-1,\, N-1}, \tag{4}$$

where $\mathbf{h}$ is $N \times N$ matrix of the Hamiltonian $\hat{h} = \widehat{H}\hbar\omega$ in the basis (2). This matrix is the sum of the matrix $\mathbf{V}$ of the potential-energy operator $\widehat{V}/\hbar\omega$ (14) and the tridiagonal matrix $\mathbf{T}$ of the kinetic-energy operator $\widehat{T}/\hbar\omega$. The nonzero elements of $\mathbf{T}$ are

$$\begin{aligned} T^l_{n,n} &= \tfrac{1}{2}(2n + l + \tfrac{3}{2}), \\ T^l_{n,n+1} = T^l_{n+1,n} &= -\tfrac{1}{2}\sqrt{(n+1)(n+l+\tfrac{3}{2})}. \end{aligned} \tag{5}$$

Employing the spectral decomposition representation of the matrix $\mathbf{h} = \mathbf{Z\Lambda Z}^*$, the function $\mathcal{P}_N(\epsilon)$ can be written in the form

$$\mathcal{P}_N(\epsilon) = \sum_{j=1}^{N} \frac{Z^2_{N,\, j}}{\epsilon - \lambda_j}, \tag{6}$$

(which bears resemblance to the expression for $R-$matrix [2]). Note that the function $\mathcal{P}_N(\epsilon)$ contains necessary information for construction of potential matrix $\mathbf{V}$. In fact it's residues and poles define squared last row elements $\left\{Z^2_{N,j}\right\}$ of eigenvectors matrix $\mathbf{Z}$ and the eigenvalues $\{\lambda_j\}$ of the $\mathbf{h}$ matrix.

From the definition (4) it follows that the potentials (1) of rank N with a fixed parameter $\hbar\omega$ are phase-equivalent if their matrices $\mathbf{V}$ satisfy the relation

$$\mathbf{V}(Q) = \mathbf{Q}\mathbf{h}_0\mathbf{Q}^* - \mathbf{T}, \tag{7}$$

where $\mathbf{h}_0$ is fixed symmetric tridiagonal matrix and $\mathbf{Q}$ is an arbitrary $N \times N$ orthogonal matrix of the form

$$\mathbf{Q} = \left(\begin{array}{c|c} \boxed{\quad \overline{\mathbf{Q}} \quad} & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \\ 0 \ldots 0 & 1 \end{array} \right). \tag{8}$$

On the other hand from the equation (3) it follows that the function $\mathcal{P}_N(\epsilon)$ defines the scattering and the discrete spectrum of the system.

To invert the $s-$function on some energy interval $[0,\, E_0]$ we propose the following procedure [3]: 1) to fix the values of parameters $\hbar\omega$, N and to define the function

$$\widetilde{\mathcal{P}}_N(\epsilon) = \frac{1}{T^l_{N-1,\,N}} \frac{C^{(-)}_{N-1,\,l}(\epsilon) - s(\epsilon) C^{(+)}_{N-1,\,l}(\epsilon)}{C^{(-)}_{N,\,l}(\epsilon) - s(\epsilon) C^{(+)}_{N,\,l}(\epsilon)}; \tag{9}$$

2) to construct the rational approximants of the form (6) for the function $\widetilde{\mathcal{P}}_N(\epsilon)$ on the interval $\epsilon \in [0,\, \epsilon_0 = E_0/\hbar\omega]$.

A relationship between the elements of the method is represented below:

$$\begin{array}{ccccc} \mathbf{V}(Q) & \longleftarrow & \mathcal{P} & \longleftarrow & \widetilde{\mathcal{P}} \\ & & \uparrow & & \uparrow \\ & & \mathbf{V} & \longrightarrow & s \end{array} \tag{10}$$

To define "inner" parameters $\left\{ Z^2_{N,j},\, 0 < \lambda_j < \epsilon_0 \right\}$ well-known method in the $R-$matrix approach [2] is followed. Namely, $\left\{ Z^2_{N,j},\, 0 < \lambda_j < \epsilon_0 \right\}$ are identified with the residues and poles of the function $\widetilde{\mathcal{P}}_N(\epsilon)$. Need of "external" parameters $\left\{ Z^2_{N,j},\, \lambda_j > \epsilon_0 \text{ and/or } \lambda_j < 0 \right\}$ is caused on the one hand by normalization condition

$$\sum_{j=1}^{N} Z^2_{N,\,j} = 1, \tag{11}$$

and on the other hand by necessity of description of the bound states. Because in the presence of the bound states with the energies $\{\varepsilon_\nu < 0\}$ the function $\mathcal{P}_N(\epsilon)$ has factors $\dfrac{\epsilon - \mu_\nu}{\epsilon - \lambda_\nu}$: $\varepsilon_\nu < \lambda_\nu < \mu_\nu$, and with sufficiently large N the inequality $\mu_\nu < 0$ is correct. If there are bound states an additional requirements imposed on the parameters $\left\{ Z^2_{N,j},\, \lambda_j \right\}$ are of the form [4]

$$\mathcal{P}_N(\varepsilon_\nu) = \frac{1}{T^l_{N-1,N}} \frac{C^{(+)}_{N-1,l}(i\sqrt{-2\varepsilon_\nu})}{C^{(+)}_{N,l}(i\sqrt{-2\varepsilon_\nu})}. \tag{12}$$

A practical implementation of the method includes the choice the parameters $\hbar\omega$, N optimal combination. The parameter $\hbar\omega$ plays a crucial role in the process. An appropriate choice of $\hbar\omega$ allows to minimize the rank N of desired potential (1).

The scheme outlined above can be expanded to the elastic scattering case with taking account of two (or more) coupled channels. The potential is of the form

$$\widehat{V} = \sum_{\alpha,\,\beta\,=\,1,\,2} |\alpha\rangle\, V^{\alpha\,\beta}\, \langle\beta|\,, \tag{13}$$

where the partial potentials $V^{\alpha\,\beta}$ are defined by the expressions

$$V^{\alpha\,\beta} = \hbar\omega \sum_{n=0}^{N_\alpha-1} \sum_{n'=0}^{N_\beta-1} \left| \varphi^{(\alpha)}_n \right\rangle V^{\alpha\,\beta}_{n,\,n'} \left\langle \varphi^{(\beta)}_{n'} \right|. \tag{14}$$

The functions $\widetilde{\mathcal{P}}_N$, $\mathcal{P}_N$ and s became the matrix of order two.

For instance the elements of the $\mathcal{P}-$matrix are defined by following equations:

$$\mathcal{P}_{11}(\epsilon) = \sum_{j=1}^{\mathcal{N}} \frac{Z^2_{N_1,\,j}}{\epsilon - \lambda_j}, \tag{15}$$

$$\mathcal{P}_{22}(\epsilon) = \sum_{j=1}^{\mathcal{N}} \frac{Z^2_{\mathcal{N},\,j}}{\epsilon - \lambda_j}, \tag{16}$$

$$\mathcal{P}_{12}(\epsilon) = \mathcal{P}_{21}(\epsilon) = \sum_{j=1}^{\mathcal{N}} \frac{Z_{N_1,\,j}\, Z_{\mathcal{N},\,j}}{\epsilon - \lambda_j}. \tag{17}$$

$Z_{i,\,j}$ are the components of eigenvectors matrix $\mathbf{Z}$ of the Hamiltonian $\mathcal{N}\times\mathcal{N}$ ($\mathcal{N} = N_1 + N_2$) matrix $\mathbf{h}$, calculated in the basis $\left\{\varphi_n^{(\alpha)};\, \alpha = 1,\, 2;\, n = \overline{0,\, N_\alpha - 1}\right\}$.

As an example of the method, we invert $S-$matrix of neutron-proton elastic scattering. np system is in the triplet spin state with total angular momentum $J = 1$. The nonzero off-diagonal elements $V_{nn'}^{\alpha\neq\beta}$ of the potential matrix (14) imitate the tensor interaction connecting the partial amplitudes corresponding to orbital momenta $l_1 = 0$ and $l_2 = 2$. The results corresponding to the parameters given in the table are displayed in the figures (solid line); the crosses correspond to the data [5].

Table of the parameters $\{\lambda_j\}$, $\{Z^2_{N_1,\,j}\}$, $\{Z^2_{\mathcal{N},\,j}\}$ and signs $\{s_j = sign(Z_{N_1,\,j}\, Z_{\mathcal{N},\,j})\}$ of the $\mathcal{P}-$matrix of the potential (14) with $\hbar\omega = 62\ MeV$, $l_1 = 0$, $l_2 = 2$, $N_1 = N_2 = 4$.

j	λ_j	$Z^2_{N_1,\,j}$	$Z^2_{\mathcal{N},\,j}$	s_j
1	.00719953	.00875	.0000536257	+
2	.799503	.0186411	.0543272	−
3	.838921	.0953768	.0119322	+
4	2.10551	.00280480	.167453	−
5	2.51864	.329940	.00278319	+
6	4.0265	.00397203	.290857	−
7	5.25	.539765	.00319449	+
8	6.85	.00075	.469400	−

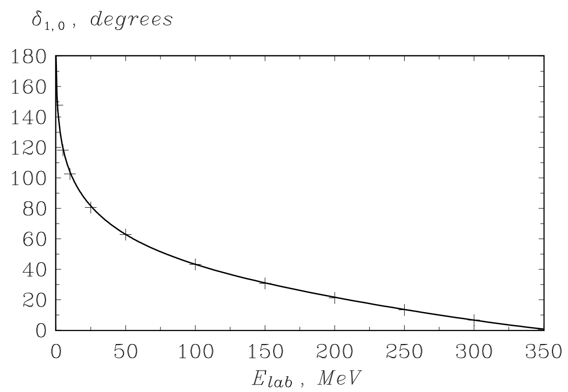

Figure 1:

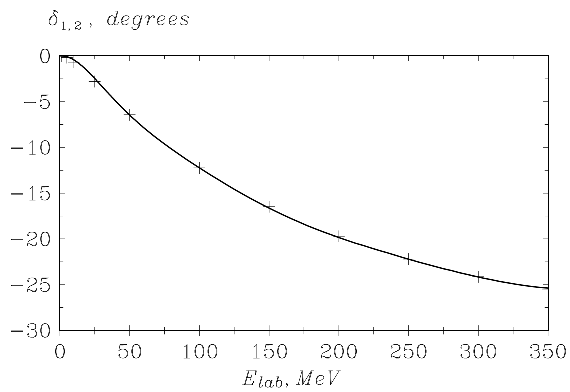

Figure 2:

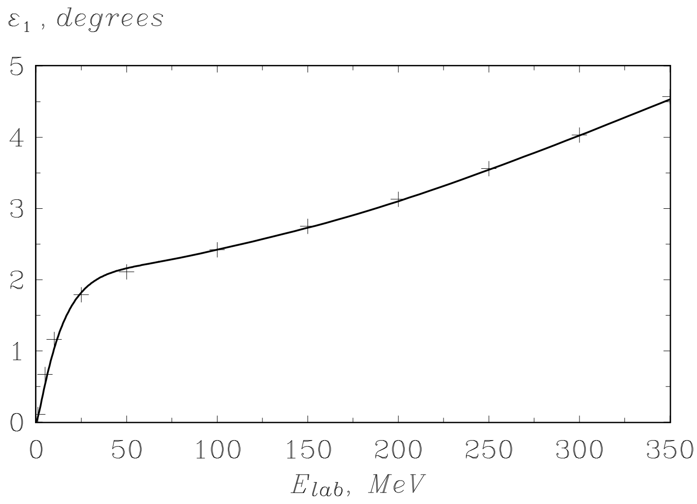

Figure 3:

References

[1] H. A. Yamani, L. Fishman, J. Math. Phys. 16 (1975) 410.

[2] B.N. Zakhariev, A.A. Suzko. Direct and inverse problems. In: Potentials in quantum scattering. 2-nd ed., Berlin, Heidelberg, New York: Springer-Verlag, 1990.

[3] S. A. Zaitsev, Theor. Math. Phys. 115 (1998) 575.

[4] S. A. Zaitsev, Yu. F. Smirnov, A. M. Shirokov, Theor. Math. Phys. 117 (1998) 1291.

[5] V. G. J. Stoks, R. A. M. Klomp, C. P. F. Terheggen, and J. J. de Swart Phys. Rev. C49 (1994) 2950.